Utah State Math Contest 2003
Preliminary Exam with Solutions, Grades 7-9.

Problem 1. The sum of the solutions to the equation $x^2 + |2x - 16| = 19$ is

(a) -4 (b) -2 (c) 0 (d) 2 (e) 4

Solution. Since

$$|2x-16| = \begin{cases} 2x-16 & \text{if } \ 2x-16 \geq 0 \\ -2x+16 & \text{if } \ 2x-16 < 0 \end{cases}$$

we have

$$\begin{cases} x^2+2x-16 = 19 \\ \quad \text{if} \quad x \geq 8 \end{cases} \qquad \text{or} \qquad \begin{cases} x^2-2x+16 = 19 \\ \quad \text{if} \quad x < 8 \end{cases}.$$

That simplifies to

$$\begin{cases} (x+7)(x-5) = 0 \\ \quad \text{if} \quad x \geq 8 \end{cases} \qquad \text{or} \qquad \begin{cases} (x-3)(x+1) = 0 \\ \quad \text{if} \quad x < 8 \end{cases}.$$

The first case gives no solutions and the second case gives $x = 3, x = -1$. Thus the sum of all solutions is 2.

Problem 2. Betty has 60% as many dimes as she has nickels. The combined value of the two is \$3.85 . How many more nickels than dimes does she have?

(a) 12 (b) 13 (c) 14 (d) 15 (e) 16

Solution. Let d be the number of dimes and n be the number of nickels that Betty has in her possession. We obtain the following system of equations:

$$\begin{cases} d = .6n \\ .10d + .05n = 3.85 \end{cases} \qquad \text{substitute} \qquad \begin{cases} d = .6n \\ .11n = 3.85 \end{cases}.$$

Thus, $n = \dfrac{3.85}{.11} = 35$ and $d = .6 \cdot 35 = 21$. Betty has 14 more nickels than dimes.
Check: 60% of 35 is 21. Also, 35 nickels give \$1.75; 21 dimes is \$2.10, which sums up to \$3.85.

Problem 3. Which value is the solution of this equation closest to?

$$\frac{1}{5}\left(4x + \frac{25}{3}\right) - \frac{1}{3}\left(\frac{3x}{5} - 2\right) = -\frac{2}{3}$$

(a) -6.0 (b) -3.0 (c) 0 (d) 3.0 (e) 6.0

Solution.

$$\frac{1}{5}\left(4x + \frac{25}{3}\right) - \frac{1}{3}\left(\frac{3x}{5} - 2\right) = -\frac{2}{3} \qquad \frac{4x}{5} - \frac{x}{5} = -\frac{2}{3} - \frac{2}{3} - \frac{5}{3}$$

$$\frac{3x}{5} = -\frac{9}{3}; \qquad x = -\frac{9}{3} \cdot \frac{5}{3} = -5$$

The correct answer is (a).

Problem 4. Right now two clocks agree but the first clock loses 3.4 seconds every hour and the second gains 2.6 seconds every hour. How many hours will it be until the clocks again show the same time?

(a) 14,400 (b) 12,600 (c) 7,200 (d) 6,300 (e) none of these

Solution. Let x be the number of hours until the clocks show again the same hour.
Each hour, the faster clock gains $3.4+2.6=6$ seconds over the slower clock; so, after x hours, they differ by $6x$ seconds. For example, after $x=150$ hours the two clocks will differ by 900 seconds; that is, by 15 minutes.
The clocks will show the same time when the difference of $6x$ seconds equals 12 hours; that is $12\cdot 60\cdot 60$ seconds. (Since 6 divides $12\cdot 60\cdot 60$ we don't need to take multiples of 12 hours.) We obtain

$$6x \text{ seconds} = 12\cdot 60\cdot 60 \text{ seconds}; \qquad x = 2\cdot 60\cdot 60 = 7,200.$$

The clocks will show again the same time after $7,200$ hours.
Additional check: Suppose that the clocks show also a date and a.m./p.m. Assume they both show the present time as Jan 1, 00:00 a.m. After $7,200$ hours, that is, after 300 days, the true time would be October 27, 00:00 a.m. The slower clock would show $7,200\cdot 3.4\text{sec} = \frac{7,200\cdot 3.4}{60\cdot 60}\text{h} = 6\text{h}48\text{min}$ less time passing. Thus the first clock would show Oct 26, 5:12p.m.
Similarly (do it yourself!), the second clock would show Oct 27, 5:12 a.m. Using 12 hour clocks we would just see 5:12 on both clocks.

Problem 5. If x^2+2x-1 is a factor of x^3+Ax^2+Bx-1 then $A+B$ equals:

(a) -2 (b) 0 (c) 2 (d) 4 (e) none of these

Solution 1. If x^2+2x-1 is a factor of x^3+Ax^2+Bx-1, then x^3+Ax^2+Bx-1 can be written as a product of that factor and $kx+m$ for some k,m. One way to obtain k,m is to multiply the two factors and compare the coefficients of the two polynomials of third degree (because two polynomials are equal if and only if they have the same degree and all corresponding coefficients equal). However, here it suffices to compare coefficients of x^3 and that of the free term (so at this stage we can avoid the hard work of multiplying factors).
We get $1=1\cdot k$ and $-1=(-1)\cdot m$; so the other factor is $x+1$. Now, multiplying factors is really necessary (oh!), so we get:

$$x^3+Ax^2+Bx-1 = (x^2+2x-1)(x+1) = x^3+3x^2+x-1; \qquad A=3, B=1.$$

Thus $A+B=4$.
Solution 2. Divide x^2+2x-1 into x^3+Ax^2+Bx-1 and make use of the fact that it must divide with no remainder.

Problem 6. What is the sum of the solutions of the following equation?

$$2 \cdot 4^{4t^2+11t+8} = 8^{4t^2+12t+9}$$

(a) -7/2 (b) -3/2 (c) 3/2 (d) 7/2 (e) none of these

Solution. Since all of $2, 4$ and 8 are powers of 2, we can rewrite the original equation with the common base 2 and then compare the exponents. This is a correct procedure because the exponential function $f(x) = a^x$ is a one-to-one function for any $a > 0,\ a \neq 1$; so, equal outputs can be obtained only for equal inputs. In other words, $2^u = 2^w$ if and only if $u = w$.
Therefore we obtain

$$2 \cdot 4^{4t^2+11t+8} = 8^{4t^2+12t+9}$$
$$2^{2\cdot(4t^2+11t+8)+1} = 2^{3\cdot(4t^2+12t+9)}$$
$$2 \cdot (4t^2 + 11t + 8) + 1 = 3 \cdot (4t^2 + 12t + 9)$$

Solving above quadratic equation leads to

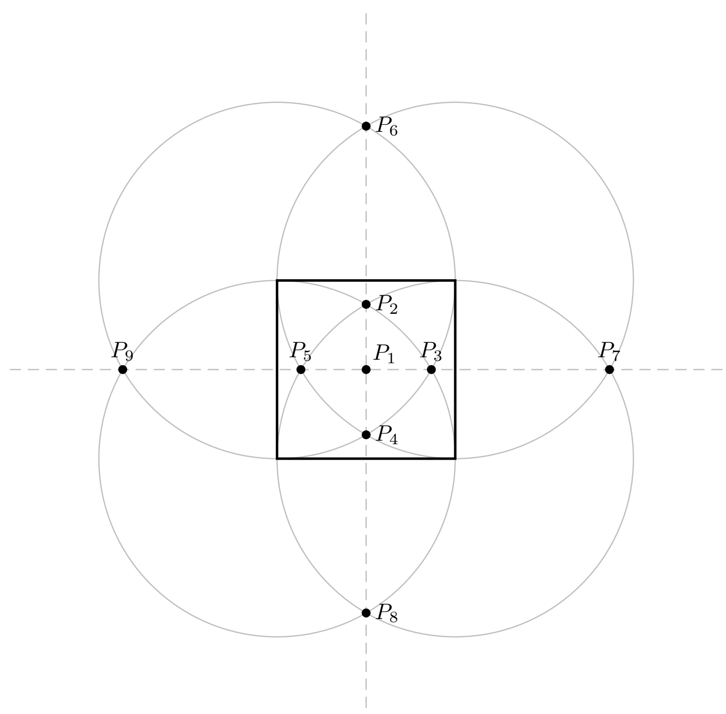
$$2t^2 + 7t + 5 = 0$$
$$(2t + 5)(t + 1) = 0$$

which gives $t = -1, t = -\frac{5}{2}$. Therefore the sum of the solutions is $-7/2$.

Problem 7. $ABCD$ is a square. How many points P are there in the plane of the square that have the property that PAB, PBC, PCD, PDA are all isosceles triangles?

(a) 1 (b) 4 (c) 5 (d) 9 (e) infinitely many

Solution. Suppose that point P lies on the plane of the square $ABCD$ and is such that $\triangle PAB$, $\triangle PBC$, $\triangle PCD$, $\triangle PDA$ are all isosceles triangles.
If a side of the square, say side AB, is a base of one of such a triangle, then point P lies on the perpendicular bisector of side AB. On the picture below the perpendicular bisectors of the square's sides are labeled as as lines l, k.
If a side AB is one of the equal sides in such a triangle, then point P lies on one of the circles centered at A or B, with radius equal to the length of the side of the square.
As illustrated on the picture, there are 9 points of intersection of such lines and circles.



Problem 8. If each angle of a regular n-gon (regular polygon with n sides) is $175°$, what is the value of n?

(a) 6 (b) 12 (c) 24 (d) 56 (e) 72

Solution 1. To solve this problem we need first to derive a formula for the angle sum of a regular n-gon (as a function of n). Pick any vertex and connect it to all the other vertices forming $n-2$ triangles, and hence showing that the sum of all the angles in the $n-$gon is equal to $(n-2)180°$. Make sure to draw a picture to convince yourself. Therefore

$$\text{Angle sum of regular } n\text{-gon is } \ (n-2)180°.$$

In a regular n-gon all angles have equal measure. Thus

$$(n-2)180° = 175°n$$

which gives $n = 72$.
Solution 2. One can make use of the fact that every polygon has an exterior angle sum of $360°$. Our polygon has exterior angles of $180° - 175° = 5°$ degrees, so it must have $n = 360°/5° = 72$ angles and sides.

Problem 9. An ant starts to climb a can that has the shape of a right circular cylinder with a height of 15 inches.Instead of climbing straight up the ant crawls on an angle so that it spirals up like the stripe on a barbershop pole. If the radius of the can is $6/\pi$ inches and the ant goes around exactly 3 times until it reaches the top, how far did the ant crawl along the side of the can?

(a) 36 (b) 39 (c) 42 (d) 45 (e) none of these

Solution. Imagine yourself cutting the can straight up and unfolding it on a table. You would get a rectangle with sides $12 = 2 \cdot \pi \cdot \frac{6}{\pi}$ and 15 inches. Imagine the trace made by the ant. The first time around the ant goes along a straight line segment from the lower corner of the rectangle towards the opposite longer side and reaches a height of 5 inches. Thus it crawls a distance of $\sqrt{12^2 + 5^2} = 13$ inches. It crawls the same distance during the second and third rounds, so altogether it crawls 39 inches.

Problem 10. Line L_1 passes through the intersection point of lines L_2 and L_3 that correspond to $3x - 4y = -12$ and $x + 2y = 10$, respectively. L_1 is also perpendicular to line L_3. What is the y-intercept of line L_1?

(a) -2 (b) -1 (c) 0 (d) 1 (e) 2

Solution. Since the slope of L_3 is $-1/2$ and since the slopes of perpendicular lines are negative reciprocals of one another, we conclude that the slope of L_1 is 2. Thus the slope-intercept equation of L_1 is $y = 2x + b$. To calculate b we need the coordinates of a point belonging to line L_1. We will use coordinates of P, the point of intersection of lines L_2 and L_3, which can be found by solving the following system of equations:

$$\begin{cases} 3x & - & 4y & = & -12 \\ x & + & 2y & = & 10 \end{cases}$$

The result is $x = \dfrac{8}{5}$ and $y = \dfrac{21}{5}$. Substituting coordinates of point P into $y = 2x + b$ yields $b = 1$.
Additional check. If the point P is really a point of intersection of all three lines, its coordinates must work in all of the equations.

$$\begin{cases} 3 \cdot \frac{8}{5} & - & 4 \cdot \frac{21}{5} & = & -12 \\ \frac{8}{5} & + & 2 \cdot \frac{21}{5} & = & 10 \\ 2 \cdot \frac{8}{5} & + & 1 & = & \frac{21}{5} \end{cases}$$

Problem 11. What is the largest difference there can be between two three digit numbers (at least 100) that have the same digits (possibly in different orders)?

(a) 799 (b) 800 (c) 802 (d) 899 (e) none of these

Solution. The difference $999 - 100 = 899$ is the largest for any two three digit numbers (without the restriction of the same digits). Obtaining 800 or more requires $9xy$ for the larger number and $1zw$ for the smaller number. In this case one of x, y must be 1 and one of z, w must be 9. The remaining digits must be the same.
Let's see if we can obtain a difference of at least 800 (meaning no "borrowing" from 9 in $9xy$ is allowed). Well, $910 - 109$ gives a difference of 801. By considering the nine other cases what x can be, we see that 801 is the largest possible difference. (For example, if $x = 2$ then either $921 - 129 = 792$; or $921 - 192 = 729$; both giving a smaller difference than 801.)
Therefore we should mark (e) as the correct answer.

Problem 12. A motorboat that goes 9 miles per hour in still water traveled downstream for 36 miles. Then it went back upstream to its starting point. If the entire trip required 9 hours, which one of the following is closest to the speed of the stream?

(a) 2.2 mph (b) 2.5 mph (c) 2.8 mph (d) 3.1 mph (e) 3.4 mph

Solution. Let v be the speed of the stream in miles per hour and let t be the time of the trip downstream in hours. Then
$9 - v$ miles per hour is the speed of the motorboat going downstream;
$9 + v$ miles per hour is the speed of the motorboat going upstream;
$9 - t$ hours is the time of the trip upstream.
Thus $\begin{cases} (9-v)(9-t) = 36 & \textit{(The distance upstream is 36 miles.)} \\ (v+9)t = 36 & \textit{(The distance downstream is 36 miles.)} \end{cases}$
Solving for t from the second and substituting into the first yields

$$36 = (9-v)(9-t) = (9-v)\left(9 - \frac{36}{v+9}\right) = 9(9-v)\left(\frac{v+5}{v+9}\right)$$

Then, clearing the denominator and taking the positive root of the resulting equation yields the speed of the stream $v = 3$ miles per hour; then $t = 3$ hours if you also solve for t.
Thus (d) is the correct answer.
Check. Downstream the motorboat travels with the speed (relative to a shore) of 12 miles per hour for 3 hours (giving the desired 36 miles). Upstream the trip takes twice as long, 6 hours, with the speed of 6 miles per hour (again, we obtain 36 miles as the distance traveled.)

Problem 13. What is the sum of the digits of the smallest positive integer that is both greater than 1, and leaves a remainder of 1 when divided by either 5,6,7 or 8?

(a) 5 (b) 9 (c) 11 (d) 13 (e) 16

Solution. The smallest number divisible by $5, 6, 7$ and 8 is the least common multiple of these numbers which is $5 \cdot 2 \cdot 3 \cdot 7 \cdot 4 = 840$. To get a remainder of 1 we should add one, which gives the sum of the digits 13.
Check. $\frac{841}{5} = \frac{840+1}{5} = \frac{840}{5} + \frac{1}{5} = \frac{5 \cdot 2 \cdot 3 \cdot 7 \cdot 4}{5} + \frac{1}{5} = 2 \cdot 3 \cdot 7 \cdot 4 + \frac{1}{5}.$
Similarly for the division by $6, 7$ and 8.

Problem 14. One root of the equation $8x^2 - 2x + c = 0$ is $x = -1/2$. What is the other root?

(a) 1/2 (b) 3/4 (c) 3/2 (d) 2 (e) none of these

Solution 1. To calculate c evaluate the polynomial at $x = -1/2$. It must give an output of zero since $-1/2$ is a solution.

$$8\left(\frac{-1}{2}\right)^2 - 2\frac{-1}{2} + c = 0, \qquad \text{from which } c = -3$$

Now, solve the equation $8x^2 - 2x - 3 = 0$ either by factoring (works in easy cases) or by using the quadratic formula (always works!) to conclude that the other root of the equation is $x = 3/4$.

Solution 2. The graph of a quadratic function $f(x) = ax^2+bx+c$ is a parabola, which is symmetric about the vertical line passing through the vertex. The equation of the line is associated with the x coordinate of the vertex, $x = -\dfrac{b}{2a}$; in our case, $x = -\dfrac{1}{8}$. The second solution must be symmetric with the first through $x = -\dfrac{1}{8}$.

Thus the other root of the equation is given by

$$\frac{1}{8} + \left(\frac{1}{8} - \left(-\frac{1}{2}\right)\right) = \frac{3}{4}.$$

Problem 15. In a survey of 75 people about brands A, B and C, 31 liked brand A, 33 liked B, and 41 liked C. These include 8 who liked only A, 3 who liked A and B but not C, 12 who liked only B and 14 who liked only C. How many of the people surveyed did not like any of the three brands?

(a) 6 (b) 9 (c)11 (d)12 (e)none of these

Solution. In counting the number of elements in a set we need to make sure that all elements are counted exactly once. Note that a statement "person likes brand C" doesn't exclude that such a person likes or dislikes the other brands.

We can partition the union of the three sets into four disjoint (nonoverlapping) pieces, as shown on the Venn diagram below. Thus,

(the number of people who like at least one of the three brands) =

(the number of people who like C) + (the number of people who only A) + (the number of people who like only B) + (the number of people who both A and B, but not C) = 41 + 8 + 12 +3 =64.

The number of people who didn't like any brand is 75-64=11.