2003 STATE MATH CONTEST

Grades 10 – 12

1. The perimeters of a square and a regular hexagon are equal. What is the ratio of the area of the hexagon to the area of the square?

   (a) $\frac{2\sqrt{3}}{3}$ (b) $\sqrt{3}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{\sqrt{2}}{3}$ (e) none of these

2. What is the number of subsets of the set $\{a, b, c, d, e, f\}$ that contain at least one vowel?

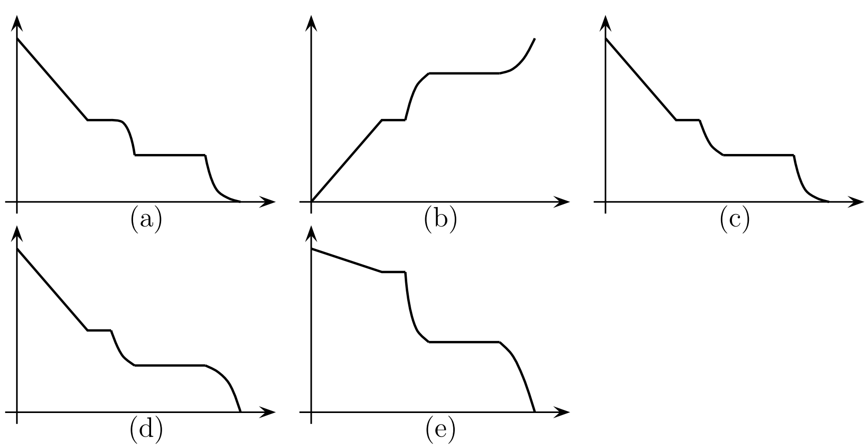
   (a) 54 (b) 48 (c) 32 (d) 36 (e) 64

3. What is the sum of the solutions of the following equation?

$$\log_3(9^x + 6) - \log_3(4 \cdot 3^x - 7) = 1$$

   (a) 2 (b) -3 (c) 3 (d) 5 (e) none of these

4. As you mow a lawn, the area of uncut grass decreases. Suppose that you start mowing after breakfast and cut about half the grass at a constant rate before taking a 20 minute break. Then you start again cutting at a slower and slower pace until taking an hour break for lunch. After lunch you are so eager to finish that you mow faster and faster until you finish. Which of the graphs best represents the amount of uncut grass in the yard as a function of time?



5. A strictly increasing geometric sequence starts with 5. An arithmetic sequence also starts with 5 and happens to have its fourth and 16th element equal to the geometric sequence's third and fifth element, respectively. What is the sum of the first ten terms of the arithmetic sequence?

(a) 50 (b) 275 (c) 330 (d) 640 (e) 5115

6. The populations (P_A and P_B) of states A and B grow according to $P_A = 3e^{0.05t}$ and $P_B = 5e^{0.03t}$, where t is the number of years from now and population is in millions. In how many years will state A have twice the population of state B?

(a) $\dfrac{\ln(5/6)}{0.02}$ (b) $\dfrac{\ln(3/10)}{0.02}$ (c) $\dfrac{\ln(3/5)}{0.02}$ (d) $\dfrac{\ln(10/3)}{0.02}$ (e) $\dfrac{\ln(15/2)}{0.02}$

7. What is the value of $\sqrt{5+2\sqrt{6}} - \sqrt{5-2\sqrt{6}}$?

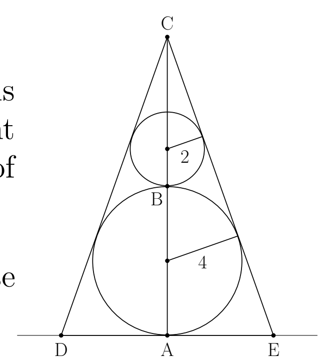
(a) $\sqrt{5}$ (b) $\sqrt{6}$ (c) $\sqrt{7}$ (d) $\sqrt{8}$ (e) $\sqrt{10}$

8. Arthur's wife picks him up at the train station and drives him home every Friday. One Friday Arthur catches the early train, arrives 90 minutes early, and starts walking home. His wife, who left home at the usual time to pick him up, meets him on the way home. They arrive home 20 minutes earlier than normal. How many minutes had Arthur been walking before his wife picked him up?

(a) 50 (b) 60 (c) 70 (d) 80 (e) 90

9. A circle of radius 4 touches a horizontal line at A. A second circle of radius 2 is balanced on top of the first one touching it at B. Two of the common tangent lines to the circles now form a triangle CDE (see figure). What is the length of DE?

(a) $2\sqrt{24}$ (b) 10 (c) $2\sqrt{32}$ (d) 12 (e) none of these



10. The sides of a triangle are the roots of $x^3 - 12x^2 + 47x - 60$, and they are all natural numbers. What is the area of the triangle?

(a) 8 (b) 6 (c) 5 (d) 7 (e) 9

11. Suppose there are six cards, each of which is colored red, yellow or blue on one side. The other side of each card has one of the symbols $\circ$, $\triangle$, or $*$ on it. Consider this statement: "Every yellow card has a $*$ on the other side." To prove or disprove the statement, which of the following card(s) must be turned over and checked?

Red	$\bigcirc$	Yellow	$\triangle$	Blue	$*$
1	2	3	4	5	6

(a) card (3) only
(b) cards (3) and (6) only
(c) cards (2), (3) and (4) only
(d) cards (2), (3) and (6) only
(e) cards (2), (3), (4), and (6) only

12. Let w, x, y, and z be natural numbers. If $w|y$ (w divides evenly into y) and $x|z$ then which of the following are always true?

(i) $wx|yz$ (ii) $(w+x)|(y+z)$ (iii) $w|yz$ (iv) $wx|xy$ (v) $x^w|z^y$

(a) all of these
(b) all but (ii)
(c) all but (v)
(d) only (i), (iii) and (iv)
(e) only (i)

13. We define a number to be of type α if it can be obtained as the final result (of step 4) of the following process:

1. Start with a 4-digit number ($ABCD$) in which the second digit from the left is 0 ($B = 0$); for example, 3021.
2. Subtract the sum of the digits from the number. (In the example, 3021-6=3015.)
3. Divide the result by 9. (In the example, 3015/9= 335)
4. Subtract $99 \times A$ from the result (in the example: $335 - (99 \times 3) = 38$.)

The example shows that 38 is of type α. Which is the number closest to 38 that is NOT of type α ?

(a) 34 (b) 35 (c) 36 (d) 44 (e) 46

14. The determinant of the matrix

$$\begin{bmatrix} x+7 & x+9 & -2 \\ 8 & x+4 & -4 \\ 2 & 3 & -1 \end{bmatrix}$$

is 0 for two values of x. What is the sum of these two values?

(a) -5 (b) -3 (c) 0 (d) 3 (e) 5

15. If $w < x$ and $y < z$ with $w, x, y, z \neq 0$ then which of the following must be true for every possible value of w, x, y, z?

(i) $\frac{1}{w} > \frac{1}{x}$

(ii) $wy < xz$

(iii) $w + y < x + z$

(iv) if $z < w$ then $y < x$.

(a) all are true (b) none are true (c) only (ii) and (iii) are true

(d) only (i), (iii) and (iv) are true (e) only (iii) and (iv) are true

16. What is the measure x of the smallest positive angle in radians, such that $\sin(x \text{ radians}) = \sin(x \text{ degrees})$?

(a) $\frac{\pi + 180}{180}$ (b) $\frac{\pi + 180}{\pi}$ (c) $\frac{\pi}{180 + \pi}$ (d) $\frac{180}{180 + \pi}$ (e) $\frac{180\pi}{180 + \pi}$

17. Region R lies below the parabola $y = 1 - x^2$, and above the parabola $y = x^2 - 1$, left of the parabola $x = 1 - y^2$ and to the right of the parabola $x = y^2 - 1$. How many lines of symmetry does R have?

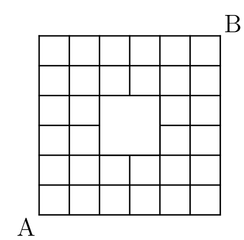
(a) none (b) 2 (c) 4 (d) 8 (e) 16

18. A three page manuscript contains three typing errors. Assume that the errors occurred at random, independently of one another. What is the probability that they are all on the same page?

(a) 1/8 (b) 1/9 (c) 1/12 (d) 1/27 (e) 1/6

19. How many paths of length 12 lead from A to B in the grid on the right?

(a) 256 (b) 384 (c) 412 (d) 524 (e) 725



20. One hundred pennies or ninety-five pennies and a nickel are two different ways to make change for one dollar. If one had 100 pennies, 4 nickels, 2 dimes and 3 quarters, how many ways could one make change for a dollar? (Assume one does not distinguish between which pennies, nickels, etc. were used.)

(a) 48 (b) 20 (c) 72 (d) 60 (e) 56

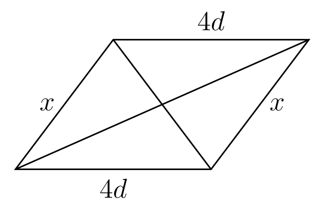
21. Players A, B and C take turns rolling a die. A starts and will win if he rolls a one. If he doesn't win, then player B rolls and wins if he rolls a two or a three. If he doesn't win either, then C rolls and wins with a four, five or six. Then A tries again, etc. and the players alternate in the order A, B, C, A, B, C, A,... until someone wins. What is the probability that C will win the game?

(a) 1/3 (b) 5/13 (c) 1/2 (d) 3/5 (e) none of these

22. What is the measure of the central angle θ in radians that cuts from the circle both an arc of length 14π inches and a corresponding sector of area 168π in^2?

(a) $\dfrac{12\pi}{7}$ (b) $\dfrac{14\pi}{5}$ (c) $\dfrac{7\pi}{12}$ (d) $\dfrac{5\pi}{14}$ (e) none of these

23. Two sides of a parallelogram (as marked on the figure) have lengths $4d$. The two diagonals are of lengths $4d$ and $6d$. How long are the other sides x of the parallelogram?



(a) $3d$ (b) $4d$ (c) $\sqrt{6}d$ (d) $5d$ (e) $\sqrt{10}d$

24. Find $\displaystyle\lim_{x\to 0} \frac{x \sin x}{\sqrt{2x^2+3}-\sqrt{3}}$.

(a) $\sqrt{5}$ (b) $\pi/\sqrt{2}$ (c) $\pi/2$ (d) $\sqrt{2}$ (e) $\sqrt{3}$

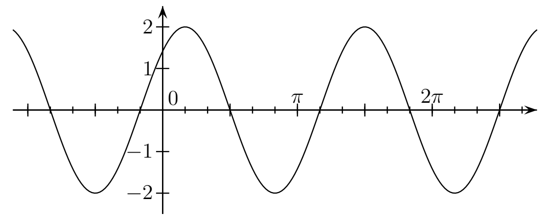
25. There is a point (p, q) on the graph of $y = x^2$ and a point (r, s) on the graph of $y = -\dfrac{8}{x}$ where both $p > 0$, $r > 0$ and the line between (p, q) and (r, s) is tangent to both curves. What is the value of $p + r$?

(a) 4 (b) 5 (c) 6 (d) 7 (e) 8

26. Given $g(x) = \displaystyle\int_1^x e^{t^2} dt$, find $\displaystyle\int_3^{x^3} e^{t^2} dt$ in terms of $g(x)$.

(a) $g(x^3) + g(3)$ (b) $g(x^3) - 3$ (c) $g(x^3) - g(3)$
(d) $g(x^3) - 3g(x)$ (e) $g^3(x) + 3$

27. Which equation is graphed at the right?



(a) $2\sin\left(\frac{1}{2}x - \frac{\pi}{4}\right)$ (b) $2\sin\left(\frac{3}{2}x + \frac{3\pi}{4}\right)$ (c) $2\sin\left(\frac{3}{2}x + \frac{\pi}{4}\right)$

(d) $2\sin\left(\frac{1}{2}x - \frac{\pi}{2}\right)$ (e) $2\sin\left(\frac{3}{2}x - \frac{\pi}{4}\right)$

28. If $f(x) = \ln(6 - x)$ and $g(x) = |x^2 - 2x - 9|$ then what is the domain of $(f \circ g)(x)$?

(a) $\{-3 < x < 1\} \cup \{3 < x < 6\}$ (b) $\{-3 < x < -1\} \cup \{3 < x < 5\}$

(c) $\{-3 < x < -1\}$ (d) $\{3 < x < 5\}$ (e) $\{-1 < x < 3\}$

29. A rectangular box of volume 405 ft^3 is to be built. The length must be twice the width. The top costs $\$2/\text{ft}^2$ to build, the sides cost $\$3/\text{ft}^2$ and the bottom costs $\$8/\text{ft}^2$. If L, W and H are the length, width and height (in feet) of the box having a minimum cost, then what is the value of $L + W + H$?

(a) 19.5 (b) 16.5 (c) 23.5 (d) 18.5 (e) none of these

30. If three fair dice are thrown, and the sum is an odd number, what is the probability that all three dice show an odd number?

(a) 1/4 (b) 1/3 (c) 1/8 (d) 1/6 (e) 1/2