2003 STATE MATH CONTEST

Solutions - Grades 7-9

1. The perimeters of a square and an equilateral triangle are equal. What is the ratio of the area of the triangle to the area of the square?

   (a) $\frac{4\sqrt{3}}{9}$ (b) $\sqrt{3}$ (c) $\frac{2\sqrt{3}}{3}$ (d) $\frac{8\sqrt{3}}{9}$ (e) none of these

   Solution:

   Let the side of the triangle be 1 (any number would do, but this will make the calculation easier). The common perimeter is 3 then, so the square has sides of length 3/4. Let's calculate the areas. The square has an area of $(3/4)^2 = 9/16$. The triangle has a height of $\sqrt{3}/2$ (which can be derived from the Pythagorean theorem if necessary) and an area of $1/2 \cdot 1 \cdot \sqrt{3}/2 = \sqrt{3}/4$. The ratio of the areas is $\sqrt{3}/4 : 9/16 = \frac{4\sqrt{3}}{9}$.

2. Given that $x + y = 1$ and $x^3 + y^3 = \dfrac{49}{4}$, find the value of $x^2 + y^2$.

   (a) 15/2 (b) 17/2 (c) 19/2 (d) 21/2 (e)23/2

   Solution 1:

   A standard solution would solve the equation system by substituting $y = 1 - x$ into the second equation, to get $x^3 - (1-x)^3 = 49/4$. After a few straightforward steps this leads to $4x^2 - 4x - 15 = 0$. This factors into $(2x + 3)(2x - 5) = 0$ or one can use the quadratic formula to get the two solutions $x = -3/2$ and $x = 5/2$. The corresponding values of y are $5/2$ and $-3/2$ respectively. This symmetry should not come as a surprise; both of the original equations were symmetric for x and y. In either case $x^2 + y^2 = (-3/2)^2 + (5/2)^2 = 17/2$.

   Solution 2:

   We don't necessarily need to find the values of x and y, only the sum of their squares. Here is a tricky way to do that.

   We cube the first equation to get

   $$1 = (x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 = (x^3 + y^3) + 3xy(x + y) = 49/4 + 3xy$$

   From this equation we get $xy = -15/4$. Why does this help us? Because $x^2 + y^2 = (x + y)^2 - 2xy = 1^2 - 2(-15/4) = 17/2$, just as in the other solution.

3. Compute $\sqrt{5+\sqrt{21}}-\sqrt{5-\sqrt{21}}$.

(a) $\sqrt{5}$ (b) $\sqrt{6}$ (c) $\sqrt{7}$ (d) $\sqrt{8}$ (e) $\sqrt{10}$

Solution 1:

Instead of the original number, let's try to compute its square:
$\left(\sqrt{5+\sqrt{21}}-\sqrt{5-\sqrt{21}}\right)^2=\left(\sqrt{5+\sqrt{21}}\right)^2-2\left(\sqrt{5+\sqrt{21}}\right)\left(\sqrt{5-\sqrt{21}}\right)+\left(\sqrt{5-\sqrt{21}}\right)^2=$
$5+\sqrt{21}-2\sqrt{(5+\sqrt{21})(5-\sqrt{21})}+5-\sqrt{21}=10-2\sqrt{25-21}=6$. Therefore the value of the original expression is $\sqrt{6}$ or $-\sqrt{6}$. We can exclude the second by noticing that the original expression is positive, the smaller number being subtracted from the larger.

Solution 2:

We can find complete squares under the square roots:

$$\sqrt{5+\sqrt{21}}-\sqrt{5-\sqrt{21}}=\sqrt{\frac{(\sqrt{7}+\sqrt{3})^2}{2}}-\sqrt{\frac{(\sqrt{7}-\sqrt{3})^2}{2}}=\frac{\sqrt{7}+\sqrt{3}}{\sqrt{2}}-\frac{\sqrt{7}-\sqrt{3}}{\sqrt{2}}=\frac{2\sqrt{3}}{\sqrt{2}}=\sqrt{6}\ ,$$

where we had to check that the squared expressions were positive before we took the square roots.

4. Arthur's wife picks him up at the train station and drives him home every Friday. One Friday Arthur catches the early train, arrives 90 minutes early, and starts walking home. His wife, who left home at the usual time to pick him up, meets him on the way home. They arrive home 20 minutes earlier than normal. How many minutes had Arthur been walking before his wife picked him up?

(a) 50 (b) 60 (c) 70 (d) 80 (e) 90

Solution:

Arthur's wife saves 20 minutes on the trip, so this must be the time it normally takes for her to drive from the place she picked up Arthur to the station and back to the pickup place. Thus after picking up Arthur, it would have taken her 10 additional minutes to arrive at the station, had she so desired. If Arthur had stayed at the station, he would have waited for 90 minutes. Since he was picked up 10 minutes earlier, he was walking for 80 minutes.

5. What is the number of elements in the smallest set that has the property that it has at least 1000 more subsets than it has elements?

(a) 9 (b) 10 (c) 50 (d) 100 (e) 1001

Solution:

The key observation is that a set with n elements has 2^n subsets. This can be shown by recognizing a pattern starting with small sets, mathematical induction, or by the following counting argument: to select a subset from a set of n elements we have to decide about each of the elements whether or not they will be selected into the subset. We have two choices for each element, a total of 2^n possible combinations. Since each combination results in a different subset, 2^n must be the number of subsets.

The rest is an easy calculation: if $n = 10$, then the number of subsets is $2^{10} = 1024$ which is at least 1000 more than the number of elements. A smaller set could have at most 9 elements and only 512 subsets. Therefore $n = 10$ is the smallest value that satisfies the conditions of the problem.

6. If $w < x$ and $y < z$ with $w, x, y, z \neq 0$ then which of the following must be true for every possible value of w, x, y, z?

(i) $\frac{1}{w} > \frac{1}{x}$

(ii) $wy < xz$

(iii) $w + y < x + z$

(iv) if $z < w$ then $y < x$.

(a) all are true (b) none are true (c) only (ii) and (iii) are true

(d) only (i), (iii) and (iv) are true (e) only (iii) and (iv) are true

Solution:

Statement (i) is not always true; if $w = -1$ and $x = 1$, then $-1 = \frac{1}{w} \not> \frac{1}{/}x = 1$.

Statement (ii) is not always true; we can see this by choosing both w and y to be -10, while x and z can be 1. Then $(-10) \cdot (-10) \not< 1 \cdot 1$.

Statement (iii) is true; we can add two inequalities side by side all the time.

Statement (iv) is also true; if $z < w$, then $y < z < w < x$ must hold as well.

The correct solution is: (e) only (iii) and (iv) are true.

7. Suppose one had six cards, each of which is colored red, yellow or blue on one side. The other side of each card has one of the symbols $\circ$, $\triangle$, or $*$ on it. Consider this statement: "Every yellow card has a $*$ on the other side." To prove or disprove the statement, which of the following card(s) must be turned over and checked?

Red	$\bigcirc$	Yellow	$\triangle$	Blue	$*$
1	2	3	4	5	6

(a) card (3) only
(b) cards (3) and (6) only
(c) cards (2), (3) and (4) only
(d) cards (2), (3) and (6) only
(e) cards (2), (3), (4), and (6) only

Solution:

Of the cards with their colored side up only the yellow card needs to be checked; the statement does not refer to any of the other colors. Of the cards with their symbol sides up cards 2 and 4 need to be checked; if their other side were yellow, the statement would become false. Card 6, however, does not need to be turned; whether or not its other side is yellow does not affect our statement. The answer is (c), cards (2),(3) and (4) need to be turned over.

8. If w, x, y, and z are natural numbers, $w|y$ (w divides y) and $x|z$ then which of the following are always true?

(i) $wx|yz$ (ii) $(w+x)|(y+z)$ (iii) $w|yz$ (iv) $wx|xy$ (v) $x^w|z^y$

(a) all of them
(b) all but (ii)
(c) all but (v)
(d) only (i), (iii) and (iv)
(e) only (i)

Solution:

A little thinking is needed to convince ourselves that (i), (iii) and (iv) are always true. To see that (ii) is false one can find a counterexample quickly, for instance $3|6$ and $5|15$ but $(3+5) \nmid (6+15)$. For (v) notice that whenever $w|y$, $w \leq y$ must also hold. Thus, we have at least as many z's as x's in the two expressions, and the divisibility holds.

The correct answer is (b); all but (ii) are true.

9. A number will be said to be type T if it can be obtained as the result in step 3 of the following process:

   1. Start with a 3-digit number (ABC); for example, 378.
   2. Subtract the sum of the digits from the number. (In the example: 378-18=360.)
   3. Divide the result by 9. (In the example: 360/9= 40)

   The example shows that 40 is of type T. What is the number closest to 40 that is NOT of type T?

   (a) 32 (b) 33 (c) 38 (d) 43 (e) 44

   Solution:

   Let's try to translate the rules into algebraic language to understand what numbers are of type T. A three digit number ABC can be written as $100A+10B+C$, where A, B and C can be any digits from 0 to 9 (with $A \neq 0$). Subtracting the sum of the digits results in $100A + 10B + C - (A + B + C) = 99A + 9B$. Dividing this by 9, we get $11A + B$ for those numbers of type T. Can every number be written in this form? No, since A and B must be single digits. The number closest to 40 that is not of this type must be $11A + 10$ for some value of A. Of the two candidates $11 \cdot 2 + 10 = 32$ and $11 \cdot 3 + 10 = 43$, the number closer to 40 is 43.

10. Which of these numbers is the greatest?

   (a) 2^{3^4} (b) 2^{4^3} (c) 3^{2^4} (d) 3^{4^2} (e)4^{3^2}

   Solution:

   By calculating the exponents quickly the numbers simplify to 2^{81}, 2^{64}, 3^{16}, 3^{16} and 4^9, respectively. Since $4^9 = 2^{18} < 2^{81}$ and $3^{16} < 4^{16} = 2^{32} < 2^{81}$, the largest of them is 2^{3^4}.

11. One hundred pennies or ninety-five pennies and a nickel are two different ways to make change for one dollar. If one had 100 pennies, 4 nickels, 2 dimes and 1 quarter, how many ways could one make change for a dollar? (Assume one does not distinguish between which pennies, nickels, etc. were used.)

   (a) 7 (b) 8 (c) 15 (d) 28 (e) 30

   Solution:

   One could list all possibilities, but we will share a more clever solution. We have enough pennies to add some to any combination of the other coins to make a dollar. How many combinations of the other coins can we get? We can have 0, 1, 2, 3, or 4 nickels (5 possibilities), 0, 1, or 2 dimes (3 possibilities) and 0 or 1 quarters (2 possibilities). These choices are independent of each other so the total number of combinations is $5 \cdot 3 \cdot 2 = 30$. (We used the fact that the all coins except the pennies do not add up to a dollar, so they can be freely combined.) We have 30 different ways to make change for a dollar.

12. Solve the system of equations

$$\begin{array}{ccccc} |x| & + & y & = & 12 \\ x & + & |y| & = & 6. \end{array}$$

What is the product of the values of x and y in the solution (x, y) ?

(a) -18 (b) 48 (c) -20 (d) 24 (e) -27

Solution:

We admit that one could guess the solution with a little bit of trying. Here is a method that requires no guessing at all. The problem with absolute values is that we can't easily manipulate them in the equations. We could get rid of absolute values if we knew whether x and y are positive or negative.

Let's first consider the situation when x is positive or zero, that is when $|x| = x$. Subtracting the first equation from the second, we will get $|y| - y = -6$, which is impossible since this difference is always positive or 0.

We found that x must be negative, so we can replace $|x|$ by $-x$. Now we add the two equations and get $y + |y| = 18$. This can only happen if $y = 9$ and then $x = -3$. The product of x and y is -27.

13. On June 30, 1983 the age of a woman was equal to the sum of the digits in the year of her birth. Which one of the following is closest to her age on June 30, 2003?

(a) 39 (b) 42 (c) 45 (d) 48 (e) 51

Solution:

First we will estimate the sum of the digits in the woman's year of birth. It is less than 28 (1+9+9+9) and therefore she was born after 1955. We could improve on this estimate, but it will do for now. The sum is also at least 16 (1+9+6+0), the smallest possible sum for the digits after 1955, so she was born before 1967. We can conclude that the woman was born in the 50's or 60's. We will now examine each of these two possibilities separately.

If she was born in $196x$ (where x is a single digit), then her age in 1983 is $23 - x = 1 + 9 + 6 + x$ which leads to $7 = 2x$. This does not provide an integer solution.

Similarly, if she was born in $195x$ then the equation for her age in 1983 gives $33 - x = 1 + 9 + 5 + x$, resulting in $x = 9$. One can check easily that the conditions are satisfied with the woman born in 1959, so she will be 44 years old in June, 2003. The closest value on the list is 45.

14. What is the units digit of 3^{2003}?

(a) 1 (b) 3 (c) 7 (d) 9 (e) none of these

Solution:

One can find a pattern of the last digits of powers of 3: they repeat the cycle 3, 9, 7, 1, 3, 9, 7,... To justify this observation we need to realize that the last digit of the next power of 3 depends ONLY on the last digit of the previous power.

Now we must figure out where does 3^{2003} fit in this repeating pattern. This depends on the remainder of 2003 when divided by 4. Since this remainder is 3, the last digit of 2^{2003} is 7, the same as the last digit of $3^3 = 27$.

15. In one county 3/4 of all females are not married, and 4/5 of all males are not married. What proportion of all people are married in the county? (Assume that every marriage is between one man and one woman.)

(a) 9/40 (b) 4/9 (c) 7/20 (d) 1/3 (e) 2/9

Solution:

Let the county have f females and m males. Since the $\frac{1}{4}f$ married females are all married to the $\frac{1}{5}m$ married males, we must have $\frac{1}{4}f = \frac{1}{5}m$. This gives $m = \frac{5}{4}f$ and the total population is $\frac{9}{4}f$ out of which the married population is $2 \cdot \frac{1}{4}f = \frac{2}{4}f$, a proportion of 2/9 of the entire population.

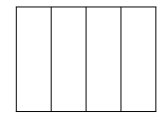
16. How many of the 900 three digit numbers are NOT divisible by either 5 or 7?

(a) 282 (b) 566 (c) 592 (d) 618 (e) none of these

Solution:

Out of the 900 three digit numbers 180 are divisible by 5, and 128 are divisible by 7. To get how many numbers are divisible by either 5 or 7, we must add these numbers, but then subtract those that we have counted twice, the numbers that are divisible by both 5 and 7. There are 26 of those; therefore 180+128-26=282 of the three digit numbers are divisible by either 5 or 7, and the remaining 618 numbers are not divisible by either 5 or 7.

17. A rancher has 600 yards of fencing for a rectangular pen divided into four equal sections (see figure). What is the largest area he can fence for his pen?



(a) 10,000 yds^2 (b) 9,000 yds^2 (c) 6,000 yds^2 (d) 22,500 yds^2 (e) none of these

Solution 1:

Let x be the width and y be the length of the pen. Then $5x + 2y = 600$ and the area of the pen is xy. From the first equation $y = 300 - 2.5x$. We need to find the value of x so that $x(300 - 2.5x)$ is as large as possible. One way to find this maximum is to rewrite the expression of the area as $9000 - \dfrac{(5x - 300)^2}{10}$ by completing the square. This is maximal if the fraction we subtract is 0 (when $x = 60$), and the largest area is 9,000 square yards.

Solution 2:

Again, let x be the width and y be the length of the pen. Then $5x + 2y = 600$ and the area of the pen is xy. At this point we will use the relationship between the arithmetic and geometric means of two positive numbers a and b: $\frac{a+b}{2} \geq \sqrt{ab}$ with equality if and only if $a = b$ (it can be proved by rearranging the inequality $(\sqrt{a} - \sqrt{b})^2 \geq 0$). We will apply this property to the numbers $5x$ and $2y$:

$$300 = \frac{5x + 2y}{2} \geq \sqrt{(5x)(2y)} = \sqrt{10xy}.$$

Squaring both sides we get $90,000 \geq 10xy$, so $xy \leq 9,000$. This maximum of 9,000 square yards can be obtained when the arithmetic mean equals to the geometric mean, that is $5x = 2y$, which combined with $5x + 2y = 600$ results in $x = 60$, $y = 150$.

18. In the system of equations $x^7y^5 = r$ and $x^4y^3 = s$ consider x, y, r, and s to be positive. When solved for x and y in terms of r and s we get $x = r^as^b$ and $y = r^cs^d$ for some a, b, c, and d. What is the sum $a + b + c + d$?

(a) 0 (b) 1 (c) 10 (d) -10 (e) none of these

Solution 1:

From the first equation $x = r^{1/7}y^{-5/7}$. Substituting this into the second equation for x and solving for y we get $y = s^7r^{-4}$. Furthermore $x = r^3s^{-5}$, and we need the sum of the exponents: $7 + (-4) + 3 + (-5) = 1$.

Solution 2:

Take the 4th power of the first equation, 7th power of the second equation to obtain $x^{28}y^{20} = r^4$ and $x^{28}y^{21} = s^7$. Dividing the second equation by the first now, we get $y = s^7r^{-4}$.

Similarly, the 3rd power of the first equation divided by the 5th power of the second provides $x = r^3s^{-5}$.

19. Given that $x(y-3) = 2y - z$, which of the following may be FALSE?

(i) if $z = 6$ and $y \neq 3$, then $x = 2$;

(ii) if $2y = z$, then $x = 0$;

(iii) if $x = 0$ and $y \neq 3$, then $z = 3$;

(iv) if $x \neq 2$, then $y = \dfrac{3x - z}{x - 2}$.

(a) only (ii) and (iii) (b) only (ii) and (iv) (c) only (iii) and (iv)

(d) only (ii), (iii) and (iv) (e) all of them

Solution:

(i) if $z = 6$, then the equation becomes $x(y-3) = 2(y-3)$, and since $y \neq 3$ we can divide both sides by $y - 3$ and get $x = 2$. The statement is always true.

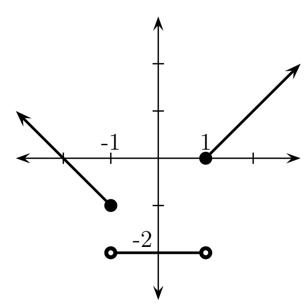
(ii) if $2y = z$, then the right hand side of the equation is 0. However, if $y = 3$ then x is not necessarily 0, the statement may be false.

(iii) if $x = 0$ and $y \neq 3$ then the left hand side is 0, but why would $z = 3$ be true?

(iv) if $x \neq 2$, then we can solve the equation for y. After rearranging we would have to divide both sides by $x - 2$, but that is legal, since $x \neq 2$. Since we get exactly the expression stated in the problem, it must always be true.

The correct answer is (a) since only (ii) and (iii) may be false.

20. Which of the following statements are TRUE about the graph of function f (see picture)?
(i) The domain (i.e. set of possible x values) of the function is all real numbers.
(ii) If $x \leq -1$ then $f(x) \geq -1$.
(iii) If $f(x) < 0$ then $|x| < 1$.



(a) all are true (b) none are true (c) only (i) and (ii) are true

(d) only (i) and (iii) are true (e) only (ii) and (iii) are true

Solution:

(i) True, there is a function value for every real number.

(ii) True, we can assume that the graph continues on the left in a straight line as indicated.

(iii) False, for example $x = -1$ corresponds to $f(x) = -1$ and $|-1| \not< 1$.

21. Which of the numbers below is closest to the value of the sum

$$\frac{1}{\sqrt{11}+\sqrt{9}}+\frac{1}{\sqrt{13}+\sqrt{11}}+\frac{1}{\sqrt{15}+\sqrt{13}}+\ldots+\frac{1}{\sqrt{79}+\sqrt{77}}+\frac{1}{\sqrt{81}+\sqrt{79}}?$$

(a) 2 (b) 2.5 (c) 3 (d) 3.5 (e) 4

Solution:

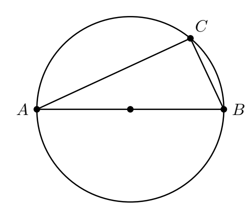
We can find this long sum exactly. A standard technique in dealing with square roots in the denominator is to multiply both the numerator and denominator of every fraction by the conjugate of the denominator, which then does not change any of the values. For example,

$$\frac{1}{\sqrt{11}+\sqrt{9}}=\frac{1}{\sqrt{11}+\sqrt{9}}\cdot\frac{\sqrt{11}-\sqrt{9}}{\sqrt{11}-\sqrt{9}}=\frac{\sqrt{11}-\sqrt{9}}{11-9}=\frac{\sqrt{11}-\sqrt{9}}{2}$$

Doing the same trick with every term, the large sum simplifies to

$$\frac{\sqrt{11}-\sqrt{9}}{2}+\frac{\sqrt{13}-\sqrt{11}}{2}+\frac{\sqrt{15}-\sqrt{13}}{2}+\ldots+\frac{\sqrt{79}-\sqrt{77}}{2}+\frac{\sqrt{81}-\sqrt{79}}{2}=\frac{\sqrt{81}-\sqrt{9}}{2}=3.$$

22. Triangle ABC is inscribed in a circle with side AB going through the center. The ratio of side AC to side BC is 2. Which of the following is closest to the ratio of the area of the circle to the area of $\triangle ABC$?



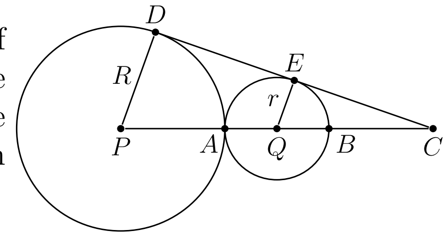
(a) $\frac{4\pi}{5}$ (b) $\frac{3\pi}{4}$ (c) π (d) $\frac{4\pi}{3}$ (e) $\frac{5\pi}{4}$

Solution:

Choose the unit so that $\overline{BC}=1$ and $\overline{AC}=2$. Since AB is a diameter of the circle, ABC is a right triangle with a right angle at vertex C. We can use the Pythagorean theorem to find that $\overline{AB}=\sqrt{5}$.

Let's compare the areas now. The area of the circle is $\left(\frac{\sqrt{5}}{2}\right)^2\pi=\frac{5\pi}{4}$, while the area of the triangle is $\frac{1}{2}\cdot 1\cdot 2=1$. The ratio of the two areas is $\frac{5\pi}{4}$.

23. A circle with radius R and center P is tangent to a second circle of radius r (where $r < R$) and center Q. The common tangent line drawn on the picture intersects line PQ at point C. What is the distance of C from the smaller circle? (i.e. find the length of BC in terms of R and r.)



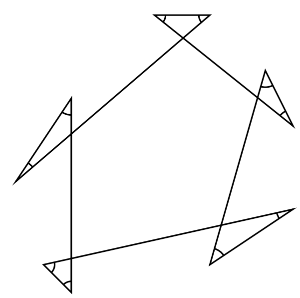
(a) $\dfrac{r^2}{R-r}$ (b) $\dfrac{2r^2}{R-r}$ (c) $\dfrac{2r^2+Rr}{r-R}$ (d) $\dfrac{2r^2}{r-R}$ (e) $\dfrac{r^2+Rr}{R-r}$

Solution:

The tangent line is perpendicular to radii PD and QE. Therefore PCD and QCE are right triangles. Since they have a common angle at C, they are also similar to each other, by the AAA similarity principle.

The rest is algebra: let's denote the unknown distance $\overline{BC}$ by x. Corresponding sides of the two similar triangles give the following ratio: $\dfrac{x+r}{x+2r+R} = \dfrac{r}{R}$ from which we can easily get $x = \dfrac{2r^2}{R-r}$.

24. Which of the following is closest to the total of the marked angles on the picture?



(a) 300° (b) 350° (c) 400° (d) 450° (e)500°

Solution 1:

The sum of all angles in the little triangles would be $5 \cdot 180° = 900°$. We get a smaller sum for the marked angles because one angle of each of the five triangles is not included in the sum. Those missing angles are identical with the interior angles of the pentagon (vertical angles), and therefore add up to 540°, the angle sum of a pentagon. The sum of the marked angles therefore is $900° - 540° = 360°$.

Solution 2:

The sum of each pair of marked angles within the same triangle is equal to the exterior angle of the angle opposite to them. Those exterior angles happen to be the exterior angles of the pentagon as well, and the sum of the exterior angles of any polygon is 360°.

25. When $200 \times 201 \times 202 \times \ldots \times 210$ is rewritten in the form of $2^n \cdot m$, where m is odd, what is the value of n?

(a) 5 (b) 6 (c) 8 (d) 10 (e) 12

Solution 1:

Every even number contributes a factor of 2 to the product, and we have 6 of them. Furthermore, every number divisible by 4 contributes with an additional factor of 2, and we have 3 of those. Similarly, we get 2 more from numbers divisible by 8, and 1 more from the only number divisible by 16. None of the numbers are divisible by any higher powers of 2. We have a total of 6+3+2+1=12 factors of two, thus $n = 12$.

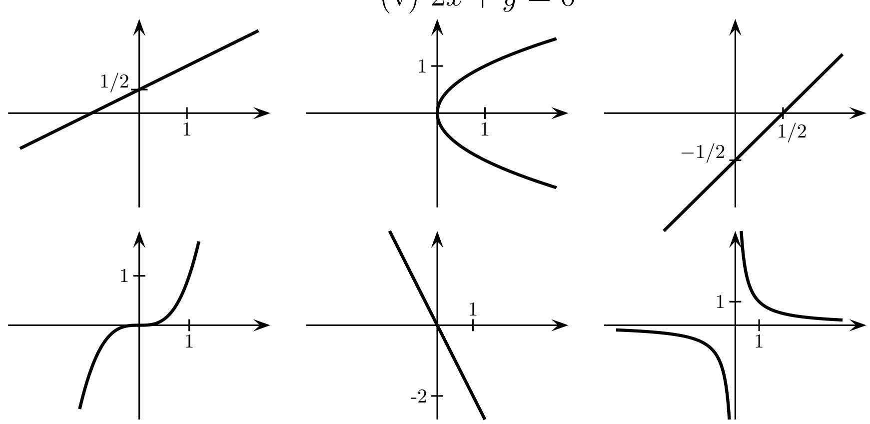
Solution 2:

We can check how many factors of 2 each number will bring into the product by dividing it by 2 as many times as possible. For example $200 = 2 \times 2 \times 2 \times 25$ brings 3 factors, 201 brings none, etc. We get a total of 3+1+2+1+4+1=12 factors of 2 again.

26. How many of the following six equations have a graph that matches one of the graphs shown?

(i) $y = x - \frac{1}{2}$ (ii) $y = x^3$ (iii) $y = \frac{1}{2}x + 1$

(iv) $xy = 1$ (v) $2x + y = 0$ (vi) $x = y^2$



(a) 2 (b) 3 (c) 4 (d) 5 (e) 6

Solution:

The first graph has no matching equation; all the other five have. The top row matches (vi) and (i), and the graphs in the bottom row match with (ii), (v) and (iv), respectively.

27. Line L is the perpendicular bisector of line segment PQ, where $P = (2, 3)$, $Q = (8, 11)$. What is the y-intercept of line L?

(a) $\frac{43}{4}$ (b) $\frac{41}{3}$ (c) $-\frac{1}{4}$ (d) $\frac{1}{3}$ (e) none of these

Solution 1:

The perpendicular bisector of the segment will go through the midpoint $(5, 7)$. The slope of segment PQ is $\frac{11-3}{8-2} = \frac{4}{3}$. The slope of a perpendicular line will have to be the negative reciprocal of that value, $-\frac{3}{4}$. The equation of the perpendicular bisector will be of the form $y = -\frac{3}{4}x + b$ where b is the y-intercept, the value we are trying to find. Since the midpoint is on the line, we can plug in its coordinates into this equation and from $7 = -\frac{3}{4} \cdot 5 + b$ we get $b = \frac{43}{4}$.

Solution 2:

The y-intercept $(0, b)$ must be of equal distance from P and Q. Using the distance formula: $(2-0)^2 + (3-b)^2 = (8-0)^2 + (11-b)^2$, which can easily be solved to give $b = \frac{43}{4}$.

28. One January there were exactly 4 Mondays and 4 Fridays. January 1 of that year was on what day of the week?

(a) Monday (b) Tuesday (c) Wednesday (d) Thursday (e) Friday

Solution:

Since January has 31 days, the first three days of the month will be repeated 5 times, the fifth time on the 29th, 30th and 31st of the month. Since we only had 4 Mondays and Fridays, these days were not among the first three days. The only way this could happen is if the month started with Tuesday, and then the first three days are Tuesday, Wednesday and Thursday. January 1 was on Tuesday that year.

29. A king gave all his castles to his 7 sons before he died. The youngest son received some number of castles, the second youngest received twice as many, the third three times as many, etc. so that the oldest son received seven times as many castles as the youngest. The queen, however, did not find this fair and ordered each son to give two of his castles to every other son that was younger than himself. After this all seven princes had the same number of castles. How many castles did the old king originally have?

(a) 98 (b) 126 (c) 105 (d) 84 (e) 112

Solution:

Assume that the smallest son received x castles originally, the next one $2x$, etc. The youngest received two castles from each of his brothers, so he will have $x + 12$ now. The oldest gave 12 castles away, and will end up with $7x - 12$. This yields the equation $x + 12 = 7x - 12$ that has one solution $x = 4$. Since every son has 16 castles at the end, the old king had $7 \times 16 = 112$ castles.

Remark:

There are a lot of other equations one could set up, all leading to the correct solution.

30. If it takes J hours for John to do a job alone, M hours for Mike to do the same job, and T hours for the two of them working together, which of the following must be true?

(a) $\dfrac{T}{M+J} = 1$ (b) $\dfrac{M+J}{T} = 1$ (c) $\dfrac{T}{M} + \dfrac{T}{J} = 1$

(d) $T(M + J) = 1$ (e) none of these

Solution:

John finishes $1/J$ of the job per hour, and Mike does $1/M$ of the job per hour. If they work together, their rate will be $1/J + 1/M = 1/T$. Multiplying both sides by T we get $\dfrac{T}{M} + \dfrac{T}{J} = 1$.