2003 STATE MATH CONTEST

Solutions - Grades 10-12

1. The perimeters of a square and a regular hexagon are equal. What is the ratio of the area of the hexagon to the area of the square?

   (a) $\dfrac{2\sqrt{3}}{3}$ (b) $\sqrt{3}$ (c) $\dfrac{\sqrt{3}}{2}$ (d) $\dfrac{\sqrt{2}}{3}$ (e) none of these

   Solution:

   Let x be the length of a side of the square and y be the length of a side of the regular hexagon. Since their perimeters are equal, this implies that $4x = 6y$, so $\dfrac{y}{x} = \dfrac{2}{3}$.

   On the other hand, the area of the square is x^2 and the area of the hexagon is $\dfrac{3\sqrt{3}y^2}{2}$ (just consider the sum of six equilateral triangles with side y). Thus the ratio of the area of the hexagon to the area of the square is

   $$\frac{3\sqrt{3}y^2}{2x^2} = \frac{3\sqrt{3}}{2}\left(\frac{y}{x}\right)^2 = \frac{3\sqrt{3}}{2}\left(\frac{2}{3}\right)^2 = \frac{2\sqrt{3}}{3}.$$

2. What is the number of subsets of the set $\{a, b, c, d, e, f\}$ that contain at least one vowel?

   (a) 54 (b) 48 (c) 32 (d) 36 (e) 64

   Solution 1:

   The total number of all subsets of $\{a, b, c, d, e, f\}$ is $2^6 = 64$. On the other hand, the number of subsets containing *no vowel* is $2^4 = 16$. Thus the number of subsets containing at least one vowel is $64 - 16 = 48$.

   Solution 2:

   We can the subsets containing a by counting all subsets of $\{b, c, d, e, f\}$, there are 2^5 of them. There are another 2^5 subsets that contain e. We have counted some subsets twice, those that contain both a and e there are 2^4 of those (subsets of $\{b, c, d, f\}$). Thus, the number of subsets containing at least one vowel is $2^5 + 2^5 - 2^4 = 48$.

3. What is the sum of the solutions of the following equation?

$$\log_3(9^x + 6) - \log_3(4 \cdot 3^x - 7) = 1$$

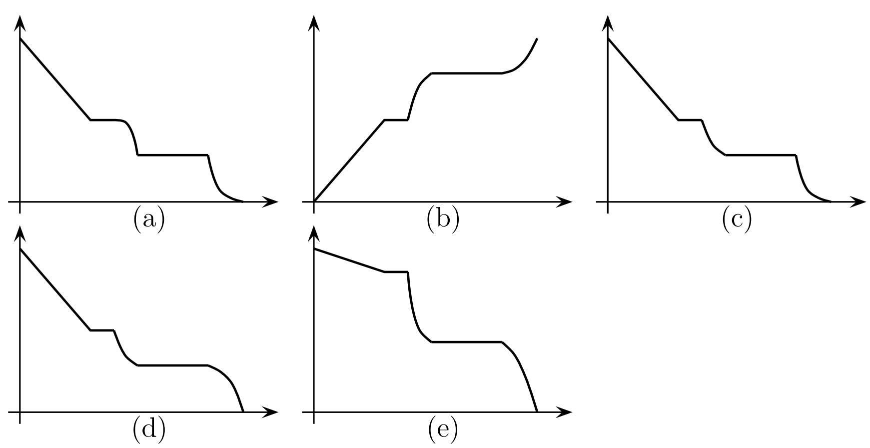
(a) 2 (b) -3 (c) 3 (d) 5 (e) none of these

Solution:

$$\begin{aligned} \log_3(9^x + 6) - \log_3(4 \cdot 3^x - 7) &= 1 \\ \log_3\left(\frac{9^x + 6}{4 \cdot 3^x - 7}\right) &= \log_3 3 \\ \frac{9^x + 6}{4 \cdot 3^x - 7} &= 3 \\ 9^x + 6 &= 12 \cdot 3^x - 21 \\ (3^x)^2 - 12(3^x) + 27 &= 0 \\ (3^x - 9)(3^x - 3) &= 0 \\ 3^x = 9 = 3^2 \ \text{ or } \ 3^x &= 3 = 3^1 \\ x = 2 \text{ or } x &= 1. \end{aligned}$$

Thus the sum of the solutions is 3.

4. As you mow a lawn, the area of uncut grass decreases. Suppose that you start mowing after breakfast and cut about half the grass at a constant rate before taking a 20 minute break. Then you start again cutting at a slower and slower pace until taking an hour break for lunch. After lunch you are so eager to finish that you mow faster and faster until you finish. Which of the graphs best represents the amount of uncut grass in the yard as a function of time?



Solution:

Graph (b) cannot be the solution because the uncut grass is a decreasing function of time. Graph (e) is wrong because the correct graph drops down to half its y-value at the first break. Graph (a) is wrong because after the first break it must get less and less steep. Graph (c) is wrong because after the second break the graph must get more and more steep. Thus the correct graph is (d).

5. A strictly increasing geometric sequence starts with 5. An arithmetic sequence also starts with 5 and happens to have its fourth and 16th element equal to the geometric sequence's third and fifth element, respectively. What is the sum of the first ten terms of the arithmetic sequence?

(a) 50 (b) 275 (c) 330 (d) 640 (e) 5115

Solution:

Let the terms of the arithmetic sequence be $a_1, a_2, a_3, ...$ and the terms of the geometric sequence be $b_1, b_2, b_3, ...$ Then, according to the conditions, $a_1 = b_1 = 5$, $a_4 = b_3$, and $a_{16} = b_5$.

On the other hand, we know that $a_n = a_1 + (n-1)d$, where d is the common difference of the arithmetic sequence, and $b_n = b_1 r^{n-1}$, where r is the common ratio of the geometric sequence.

This gives the following system of equations:

$$\begin{cases} 5+3d = 5r^2 \\ 5+15d = 5r^4 \end{cases} \to \begin{cases} (5+3d = 5r^2)\cdot 5 \\ 5+15d = 5r^4 \end{cases} \to \begin{cases} 25+15d = 25r^2 \\ 5+15d = 5r^4 \end{cases} \to \begin{cases} 20 = 25r^2 - 5r^4 \\ 5+15d = 5r^4 \end{cases}$$

$$\begin{cases} r^4 - 5r^2 - 4 = 0 \\ 5+15d = 5r^4 \end{cases} \to \begin{cases} (r^2-1)(r^2-4) = 0 \\ 5+15d = 5r^4 \end{cases} \to r = \pm 1, \pm 2$$

Since the geometric sequence is strictly increasing, the only possible value is $r = 2$. Thus $5 + 15d = 5r^4 = 80$, and so $d = 5$. We can now calculate the sum of the first ten terms of the arithmetic sequence by the formula $S_n = \left(\frac{a_1 + a_n}{2} n\right)$. Thus $S_{10} = \frac{5 + (5 + 9 \cdot 5)}{2} 10 = 275$.

6. The populations (P_A and P_B) of states A and B grow according to $P_A = 3e^{0.05t}$ and $P_B = 5e^{0.03t}$, where t is the number of years from now and population is in millions. In how many years will state A have twice the population of state B?

(a) $\frac{\ln(5/6)}{0.02}$ (b) $\frac{\ln(3/10)}{0.02}$ (c) $\frac{\ln(3/5)}{0.02}$ (d) $\frac{\ln(10/3)}{0.02}$ (e) $\frac{\ln(15/2)}{0.02}$

Solution:

"The population of state A is twice the population of state B" means that $P_A = 2P_B$. We must find the value of t when this occurs.

$$3e^{0.05t} = 2 \cdot 5e^{0.03t}$$

$$\ln(3e^{0.05t}) = \ln(10e^{0.03t})$$

$$\ln 3 + \ln e^{0.05t} = \ln 10 + \ln e^{0.03t}$$

$$\ln 3 + 0.05t = \ln 10 + 0.03t$$

$$0.02t = \ln 10 - \ln 3$$

$$t = \frac{\ln(10/3)}{0.02}.$$

7. What is the value of $\sqrt{5+2\sqrt{6}}-\sqrt{5-2\sqrt{6}}$?

(a) $\sqrt{5}$ (b) $\sqrt{6}$ (c) $\sqrt{7}$ (d) $\sqrt{8}$ (e) $\sqrt{10}$

Solution 1:

Let $x=\sqrt{5+2\sqrt{6}}-\sqrt{5-2\sqrt{6}}$. We will calculate x^2 first: $x^2=\left(\sqrt{5+2\sqrt{6}}-\sqrt{5-2\sqrt{6}}\right)^2=$ $(5+2\sqrt{6})-2\sqrt{5+2\sqrt{6}}\sqrt{5-2\sqrt{6}}+(5-2\sqrt{6})=10-2\sqrt{5^2-(2\sqrt{6})^2}=10-2\sqrt{25-24}=8$.

Thus, $x=\pm\sqrt{8}$ But $x>0$ because $\sqrt{5+2\sqrt{6}}>\sqrt{5-2\sqrt{6}}$. Therefore $x=\sqrt{8}$.

Solution 2:

$\sqrt{5+2\sqrt{6}}-\sqrt{5-2\sqrt{6}}=\sqrt{(\sqrt{3}+\sqrt{2})^2}-\sqrt{(\sqrt{3}-\sqrt{2})^2}=(\sqrt{3}+\sqrt{2})-(\sqrt{3}-\sqrt{2})=2\sqrt{2}=\sqrt{8}$.

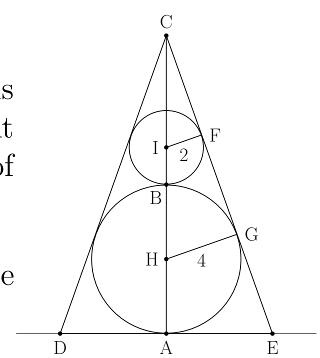
8. Arthur's wife picks him up at the train station and drives him home every Friday. One Friday Arthur catches the early train, arrives 90 minutes early, and starts walking home. His wife, who left home at the usual time to pick him up, meets him on the way home. They arrive home 20 minutes earlier than normal. How many minutes had Arthur been walking before his wife picked him up?

(a) 50 (b) 60 (c) 70 (d) 80 (e) 90

Solution:

Arthur's wife saves 20 minutes on the trip, so this must be the time it normally takes for her to drive from the place she picked up Arthur to the station and back to the pickup place. Thus after picking up Arthur, it would have taken her 10 additional minutes to arrive at the station, had she so desired. If Arthur had stayed at the station, he would have waited for 90 minutes. Since he was picked up 10 minutes earlier, he was walking for 80 minutes.

9. A circle of radius 4 touches a horizontal line at A. A second circle of radius 2 is balanced on top of the first one touching it at B. Two of the common tangent lines to the circles now form a triangle CDE (see figure). What is the length of DE?



(a) $2\sqrt{24}$ (b) 10 (c) $2\sqrt{32}$ (d) 12 (e) none of these

Solution:

The triangles $\triangle IFC$, $\triangle HGC$, and $\triangle EAC$ are all similar, right triangles, so their corresponding sides are proportional, i.e., $\frac{|IC|}{|IF|}=\frac{|HC|}{|HG|}$. But, $|IF|=2$, $|HG|=4$, and $|HI|=2+4=6$, so $|IC|=6$. Next, by the Pythagorean Theorem, $|FC|=\sqrt{32}$. Again, due to proportionality, $\frac{|AE|}{|AC|}=\frac{|IF|}{|FC|}$. However, $|AC|=|AH|+|HC|=4+12=16$ and

$\frac{|AE|}{16}=\frac{2}{\sqrt{32}}$. Thus, $|DE|=2|AE|=2\sqrt{32}$.

10. The sides of a triangle are the roots of $x^3 - 12x^2 + 47x - 60$, and they are all natural numbers. What is the area of the triangle?

(a) 8 (b) 6 (c) 5 (d) 7 (e) 9

Solution:

Let the sides of the triangle be a, b and c. Since these are integers and thus rational numbers, you may use the rational root test to try all possible divisors of 60 to obtain the factorization

$$x^3 - 12x^2 + 47x - 60 = (x-3)(x-4)(x-5).$$

Another way to find the roots of the polynomial is to write it in the form

$$x^3 - 12x^2 + 47x - 60 = (x-a)(x-b)(x-c) = x^3 - (a+b+c)x^2 + (ab+ac+bc)x - abc.$$

Comparing the coefficients of these polynomials we see that $a+b+c = 12$ and $abc = 60$. It's easy to check that $a = 3$, $b = 4$, and $c = 5$ will satisfy these conditions.

Since $3^2 + 4^2 = 5^2$, the triangle is a right triangle, so the area is $3 \cdot 4/2 = 6$.

11. Suppose there are six cards, each of which is colored red, yellow or blue on one side. The other side of each card has one of the symbols $\circ$, $\triangle$, or $*$ on it. Consider this statement: "Every yellow card has a $*$ on the other side." To prove or disprove the statement, which of the following card(s) must be turned over and checked?

Red	$\bigcirc$	Yellow	$\triangle$	Blue	$*$
1	2	3	4	5	6

(a) card (3) only (b) cards (3) and (6) only

(c) cards (2), (3) and (4) only (d) cards (2), (3) and (6) only

(e) cards (2), (3), (4), and (6) only

Solution: We must check 2 (if it's yellow, it will disprove the statement), 3 (if it's not $*$, it will disprove the statement), and 4 (if it's yellow, it will disprove the statement).

12. Let w, x, y, and z be natural numbers. If $w|y$ (w divides evenly into y) and $x|z$ then which of the following are always true?

(i) $wx|yz$ (ii) $(w+x)|(y+z)$ (iii) $w|yz$ (iv) $wx|xy$ (v) $x^w|z^y$

(a) all of these (b) all but (ii) (c) all but (v)

(d) only (i), (iii) and (iv) (e) only (i)

Solution: If a number w divides a number y, $w|y$, then there exists an integer a such that $y = aw$. Similarly, $z = bx$. Now, (i) is true since $yz = (aw)(bx) = (ab)(wx)$. (ii) is not correct: let $w = 2, y = 4, x = 3$, and $z = 9$. (iii) is correct: $yz = (aw)(bx) = (abx)w$. (iv) is correct: $xy = x(aw) = a(xw)$. (v) is correct: $z^y = (bx)^{aw} = (b^{aw}x^{(a-1)w})x^w$.

13. We define a number to be of type α if it can be obtained as the final result (of step 4) of the following process:

1. Start with a 4-digit number $(ABCD)$ in which the second digit from the left is 0 $(B = 0)$; for example, 3021.

2. Subtract the sum of the digits from the number. (In the example, 3021-6=3015.)

3. Divide the result by 9. (In the example, 3015/9= 335)

4. Subtract $99 \times A$ from the result (in the example: $335 - (99 \times 3) = 38$.)

The example shows that 38 is of type α. Which is the number closest to 38 that is NOT of type α ?

(a) 34 (b) 35 (c) 36 (d) 44 (e) 46

Solution:

A number of type α is obtained through the following procedure: first, a four-digit number $(ABCD)$ is picked with $B = 0$, so $(ABCD) = 1000A + 10C + D$ where A, C and D are integers between 0 and 9 (where $A \neq 0$). Next, the sum of digits is subtracted: $1000A + 10C + D - (A + C + D) = 999A + 9C$. Then it's divided by 9 : $(999A + 9C)/9 = 111A + C$. Finally, subtract $99A$ from the result: $111A + C - 99A = 12A + C$. This is a general formula for a number of type α. Thus, we have to find which number is closest to 38 and is *not* representable as $12A + C$. Obviously, $36 = 12 \cdot 3 + 0$ is of type α, but 35 is not of type α because $35 = 12 \cdot 2 + 11$ but C must be strictly less than 10.

14. The determinant of the matrix

$$\begin{bmatrix} x+7 & x+9 & -2 \\ 8 & x+4 & -4 \\ 2 & 3 & -1 \end{bmatrix}$$

is 0 for two values of x. What is the sum of these two values?

(a) -5 (b) -3 (c) 0 (d) 3 (e) 5

Solution:

The determinant of a matrix does not change if we add a multiple of one row to another row. Add $2\times$ the third column to the first column and $3\times$ the third column to the second column to get

$$\begin{vmatrix} x+7 & x+9 & -2 \\ 8 & x+4 & -4 \\ 2 & 3 & -1 \end{vmatrix} = \begin{vmatrix} x+3 & x+3 & -2 \\ 0 & x-8 & 4 \\ 0 & 0 & -1 \end{vmatrix} = -(x+3)(x-8) = 0$$

Thus the values which make the determinant zero are -3 and 8, so their sum is 5.

15. If $w < x$ and $y < z$ with $w, x, y, z \neq 0$ then which of the following must be true for every possible value of w, x, y, z?

(i) $\frac{1}{w} > \frac{1}{x}$

(ii) $wy < xz$

(iii) $w + y < x + z$

(iv) if $z < w$ then $y < x$.

(a) all are true (b) none are true (c) only (ii) and (iii) are true

(d) only (i), (iii) and (iv) are true (e) only (iii) and (iv) are true

Solution:

(i) is false: let $w = -1$ and $x = 1$.

(ii) is also false: let $w = 1$, $x = 3$, $y = -2$, and $z = -1$.

(iii) is true, inequalities can be added side by side.

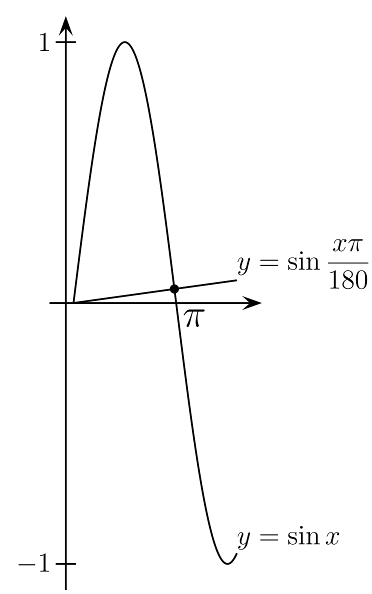
(iv) is also true: if $y < z$, $z < w$, $w < x$, then $y < x$.

Therefore (e) is the correct answer.

16. What is the measure x of the smallest positive angle in radians, such that $\sin(x \text{ radians}) = \sin(x \text{ degrees})$?

(a) $\frac{\pi + 180}{180}$ (b) $\frac{\pi + 180}{\pi}$ (c) $\frac{\pi}{180 + \pi}$ (d) $\frac{180}{180 + \pi}$ (e) $\frac{180\pi}{180 + \pi}$

Solution:



Since 1 degree $= \pi/180$ radians, converting the given equation into radians gives

$$\sin x = \sin(x\pi/180).$$

From the graphs of $\sin x$ and $\sin(x\pi/180)$, we see that x, the smallest solution, is slightly less than π, so the only possible answer is (e). To see this let $A = x$ and $B = x\pi/180$. Since both A and B are distinct and between 0 and π radians, the only way for $\sin A = \sin B$ is for $A + B = \pi$.

$$A + B = x + \frac{x\pi}{180} = \pi$$

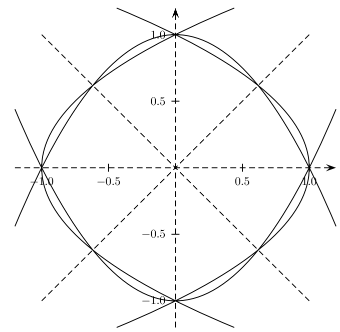
$$x = \frac{\pi}{1 + \frac{\pi}{180}} = \frac{180\pi}{180 + \pi}.$$

17. Region R lies below the parabola $y = 1 - x^2$, and above the parabola $y = x^2 - 1$, left of the parabola $x = 1 - y^2$ and to the right of the parabola $x = y^2 - 1$. How many lines of symmetry does R have?

(a) none (b) 2 (c) 4 (d) 8 (e) 16

Solution:

Making either of the substitutions $x \leftrightarrow -x$, $y \leftrightarrow -y$, $x \leftrightarrow y$, or $x \leftrightarrow -y$ gives the same boundary, so both axes and the lines $y = \pm x$ are lines of symmetry. Any other line of symmetry would also have to pass through the origin and would have to intersect one of the parabolic arcs in a point which is not the vertex or an end point of that arc. This would contradict the fact that the only line of symmetry a parabolic arc could have is the axis of the parabola. Thus, there are only 4 lines of symmetry.



18. A three page manuscript contains three typing errors. Assume that the errors occurred at random, independently of one another. What is the probability that they are all on the same page?

(a) 1/8 (b) 1/9 (c) 1/12 (d) 1/27 (e) 1/6

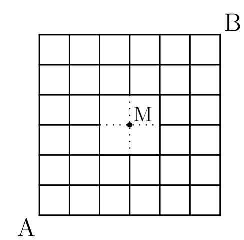
Solution:

Since the errors occur at random, the probability that an error occurs on a given page is 1/3. Because the errors are independent of each other, the probability that all three occur on a given page is $(1/3)(1/3)(1/3) = 1/27$.

But there are three pages, so the desired probability is $3 \cdot (1/27) = 1/9$.

19. How many paths of length 12 lead from A to B in the grid on the right?

(a) 256 (b) 384 (c) 412 (d) 524 (e) 725



Solution:

For the time being, assume the middle vertex of the grid is present. Then looking diagonally from A to B along a path, when you come to a vertex of the grid you can either bear left or bear right. Thus a path from A to B contains exactly 6 vertices at which you bear left and 6 vertices where you bear right; so to each path there corresponds a unique sequence such as LLRLRRRLRLLR, which, if you focus on the positions of L's, is a combination of 12 positions taken 6 at a time. The number of such combinations is ${}_{12}C_6 = \dfrac{12!}{(6!)^2} = 924$.

Now you must subtract all paths which pass through the (missing) middle grid point, M. There are ${}_6C_3 = \dfrac{6!}{(3!)^2} = 20$ paths from A to M, and the same number from M to B, so there are $20 \times 20 = 400$ illegal paths. Thus the total number of legal paths is $924 - 400 = 524$.

20. One hundred pennies or ninety-five pennies and a nickel are two different ways to make change for one dollar. If one had 100 pennies, 4 nickels, 2 dimes and 3 quarters, how many ways could one make change for a dollar? (Assume one does not distinguish between which pennies, nickels, etcẇere used.)

(a) 48 (b) 20 (c) 72 (d) 60 (e) 56

Solution:

Since there are always enough pennies to make the correct change, you must count how many ways we can choose the nickels, dimes and quarters so that their value is not greater than \$1.00. You may choose the number of nickels in 5 different ways (choose either 0, 1, 2, 3, or 4 nickels). Similarly, you can choose the number of dimes in 3 ways and the number of quarters in 4 ways. Thus there are $5 \times 3 \times 4 = 60$ ways to choose the silver coins.

The only choices that will value more than \$1 are 3 quarters, 2 dimes, and 2, 3, or 4 nickels, or 3 quarters, 1 dime, and 4 nickels. These are 4 choices, so the total number of ways of making change is $60 - 4 = 56$.

21. Players A, B and C take turns rolling a die. A starts and will win if he rolls a one. If he doesn't win, then player B rolls and wins if he rolls a two or a three. If he doesn't win either, then C rolls and wins with a four, five or six. Then A tries again, etc. and the players alternate in the order A, B, C, A, B, C, A,... until someone wins. What is the probability that C will win the game?

(a) 1/3 (b) 5/13 (c) 1/2 (d) 3/5 (e) none of these

Solution 1:

Since each roll of the die is independent of the others, the probability that player C wins on his first roll is equal to the probability that player A loses times the probability that player B loses times the probability that player C wins, which is $(5/6) \cdot (2/3) \cdot (1/2) = 5/18$. The probability that C wins on his second roll is the probability that all three players lose on their first rolls, which is the same 5/18, times the probability that player C then wins on his next roll, which is also 5/18. This probability is thus $(5/18)^2$. Continuing in this manner, the total probability that player C wins is the sum of the infinite geometric series $5/18 + (5/18)^2 + (5/18)^3 + ... = \dfrac{5/18}{1 - 5/18} = \dfrac{5}{13}$.

Solution 2:

As in the previous solution we can calculate the probability that C wins in the first round to be 5/18. Let p denote the probability that C wins the whole game. This can happen in two ways, either he wins on his first roll, or he wins later. This yields the following equation:

$P(C \text{ wins in first round}) + P(\text{ nobody wins in the first round}) \cdot P(C \text{ wins later}) = P(C \text{ wins})$

$5/18 + 5/18 \cdot p = p$, and this equation gives $p = 5/13$.

22. What is the measure of the central angle θ in radians that cuts from the circle both an arc of length 14π inches and a corresponding sector of area 168π in^2?

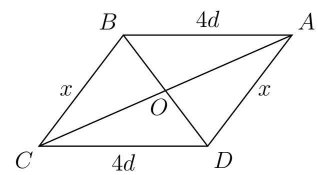
(a) $\dfrac{12\pi}{7}$ (b) $\dfrac{14\pi}{5}$ (c) $\dfrac{7\pi}{12}$ (d) $\dfrac{5\pi}{14}$ (e) none of these

Solution:

The length of an arc of a circle with radius r corresponding to the central angle θ measured in radians is $\ell = r\theta$ and the sector area is $A = r^2\theta/2$. Thus,

$$\begin{cases} r\theta = 14\pi \\ r^2\theta/2 = 168\pi \end{cases} \rightarrow \begin{cases} \theta = 14\pi/r \\ \theta = 336\pi/r^2 \end{cases} \rightarrow r = 24 \rightarrow \theta = \frac{7}{12}\pi.$$

23. Two sides of a parallelogram (as marked on the figure) have lengths $4d$. The two diagonals are of lengths $4d$ and $6d$. How long are the other sides x of the parallelogram?



(a) $3d$ (b) $4d$ (c) $\sqrt{6}d$ (d) $5d$ (e) $\sqrt{10}d$

Solution 1:

The diagonals of a parallelogram bisect each other. Thus, $|OC| = 3d$ and $|OD| = 2d$. Apply the law of cosines to the triangles $\triangle DOC$ and $\triangle BOC$, to get

$x^2 = (2d)^2 + (3d)^2 - 2(2d)(3d)\cos(\angle DOC)$ and $(4d)^2 = (2d)^2 + (3d)^2 - 2(2d)(3d)\cos(\angle BOC)$.

However, $\angle DOC$ and $\angle BOC$ are supplementary angles, so $\cos(\angle DOC) = -\cos(\angle BOC)$, so

$$\begin{cases} x^2 = (2d)^2 + (3d)^2 + 2(2d)(3d)\cos(\angle BOC) \\ (4d)^2 = (2d)^2 + (3d)^2 - 2(2d)(3d)\cos(\angle BOC) \end{cases} \rightarrow x^2 + (4d)^2 = 2\left((2d)^2 + (3d)^2\right)$$

$x^2 = 10d^2$ and $x = \sqrt{10}d$.

Solution 2:

By the parallelogram law, $2x^2 + 2(4d)^2 = (4d)^2 + (6d)^2$ from which we get $x = \sqrt{10}d$.

24. Find $\displaystyle\lim_{x\to 0} \frac{x\sin x}{\sqrt{2x^2+3} - \sqrt{3}}$.

(a) $\sqrt{5}$ (b) $\pi/\sqrt{2}$ (c) $\pi/2$ (d) $\sqrt{2}$ (e) $\sqrt{3}$

Solution 1:

$$\lim_{x\to 0} \frac{x\sin x}{\sqrt{2x^2+3} - \sqrt{3}} = \lim_{x\to 0} \frac{x\sin x}{(\sqrt{2x^2+3} - \sqrt{3})} \cdot \frac{(\sqrt{2x^2+3} + \sqrt{3})}{(\sqrt{2x^2+3} + \sqrt{3})} = \lim_{x\to 0} \frac{x\sin x(\sqrt{2x^2+3} + \sqrt{3})}{2x^2} =$$

$$= \frac{1}{2} \cdot \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0}(\sqrt{2x^2+3} + \sqrt{3}) = \frac{1}{2} \cdot 1 \cdot 2\sqrt{3} = \sqrt{3}.$$

Solution 2:

We can find the limit by applying L'Hospital's rule: $\displaystyle\lim_{x\to 0} \frac{x\sin x}{\sqrt{2x^2+3} - \sqrt{3}} =$

$$\lim_{x\to 0} \frac{\sin x + x\cos x}{\left(\frac{4x}{2\sqrt{2x^2+3}}\right)} = \lim_{x\to 0} \frac{(\sqrt{2x^2+3})(\sin x + x\cos x)}{2x} = \lim_{x\to 0} \frac{\sqrt{2x^2+3}}{2} \cdot \lim_{x\to 0}\left(\frac{\sin x}{x} + \cos x\right) = \frac{\sqrt{3}}{2} \cdot 2.$$

25. There is a point (p, q) on the graph of $y = x^2$ and a point (r, s) on the graph of $y = -\dfrac{8}{x}$ where both $p > 0$, $r > 0$ and the line between (p, q) and (r, s) is tangent to both curves. What is the value of $p + r$?

(a) 4 (b) 5 (c) 6 (d) 7 (e) 8

Solution:

According to the conditions, we have the following system of equations:

$$\begin{cases} q = p^2 \\ s = -\dfrac{8}{r} \\ 2p = \dfrac{q-s}{p-r} \\ \dfrac{8}{r^2} = \dfrac{q-s}{p-r} \end{cases} \rightarrow \begin{cases} \dfrac{8}{r^2} = \dfrac{p^2 - \left(-\dfrac{8}{r}\right)}{p-r} \\ 2p = \dfrac{8}{r^2} \end{cases} \rightarrow \begin{cases} \dfrac{8}{r^2} = \dfrac{p^2 + \dfrac{8}{r}}{p-r} \\ p = \dfrac{4}{r^2} \end{cases} \rightarrow \begin{cases} \dfrac{8}{r^2} = \dfrac{\left(\dfrac{4}{r^2}\right)^2 + \dfrac{8}{r}}{\dfrac{4}{r^2} - r} \\ p = \dfrac{4}{r^2} \end{cases}$$

$$\rightarrow \begin{cases} \dfrac{8}{r^2} = \dfrac{16 + 8r^3}{4r^2 - r^5} \\ p = \dfrac{4}{r^2} \end{cases} \rightarrow \begin{cases} 16r^5 = 16r^2 \\ p = \dfrac{4}{r^2} \end{cases} \rightarrow \begin{cases} r = 0, 1 \\ p = \dfrac{4}{r^2} \end{cases}$$

If $r = 0$ then p is undefined. If $r = 1$, then $p = \dfrac{4}{1^2} = 4$. Thus $p + r = 5$.

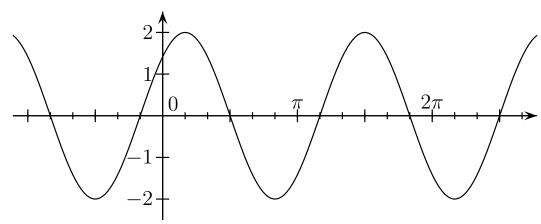
26. Given $g(x) = \displaystyle\int_1^x e^{t^2}\,dt$, find $\displaystyle\int_3^{x^3} e^{t^2}\,dt$ in terms of $g(x)$.

(a) $g(x^3) + g(3)$ (b) $g(x^3) - 3$ (c) $g(x^3) - g(3)$ (d) $g(x^3) - 3g(x)$ (e) $g^3(x) + 3$

Solution:

$\int_3^{x^3} e^{t^2}\,dt = \int_1^{x^3} e^{t^2}\,dt - \int_1^3 e^{t^2}\,dt = g(x^3) - g(3).$

27. Which equation is graphed at the right?



(a) $2\sin\left(\dfrac{1}{2}x - \dfrac{\pi}{4}\right)$ (b) $2\sin\left(\dfrac{3}{2}x + \dfrac{3\pi}{4}\right)$ (c) $2\sin\left(\dfrac{3}{2}x + \dfrac{\pi}{4}\right)$

(d) $2\sin\left(\dfrac{1}{2}x - \dfrac{\pi}{2}\right)$ (e) $2\sin\left(\dfrac{3}{2}x - \dfrac{\pi}{4}\right)$

Solution:

The equations (a), (d), and (e) are excluded, since evaluating those equations at $x = 0$ gives negative quantities, and the graph has a positive value at $x = 0$. Next, evaluating the equation (c) at $x = \pi/2$ gives a value of zero, just like on the graph. However the equation (b) does not produce a zero value at $x = \pi/2$.

28. If $f(x) = \ln(6-x)$ and $g(x) = |x^2 - 2x - 9|$ then what is the domain of $(f \circ g)(x)$?

(a) $\{-3 < x < 1\} \cup \{3 < x < 6\}$ (b) $\{-3 < x < -1\} \cup \{3 < x < 5\}$

(c) $\{-3 < x < -1\}$ (d) $\{3 < x < 5\}$ (e) $\{-1 < x < 3\}$

Solution:

Since $(f \circ g)(x) = \ln(6 - |x^2 - 2x - 9|)$ and the logarithmic function is defined only for positive numbers, we must find all x so that $h(x) = 6 - |x^2 - 2x - 9| > 0$. Now let's find all the zeros of the continuous function $h(x)$, because $h(x)$ must have a constant sign between any two consecutive zeros. $h(x) = 6 - |x^2 - 2x - 9| = 0$ yields $x^2 - 2x - 9 = \pm 6$ and solving both quadratic equations, we get $x = -3$, $x = 5$, $x = -1$, or $x = 3$ as roots.

It is easy to check, picking numbers and evaluating the function, that $h(x)$ is positive only on $(-3, -1) \cup (3, 5)$.

29. A rectangular box of volume 405 ft^3 is to be built. The length must be twice the width. The top costs $\$2/\text{ft}^2$ to build, the sides cost $\$3/\text{ft}^2$ and the bottom costs $\$8/\text{ft}^2$. If L, W and H are the length, width and height (in feet) of the box having a minimum cost, then what is the value of $L + W + H$?

(a) 19.5 (b) 16.5 (c) 23.5 (d) 18.5 (e) none of these

Solution:

We must minimize the cost C where $C = (LW)2 + (2HL + 2HW)3 + LW \cdot 8$.

We are given that $L = 2W$ and the volume $= LWH = 405$, so we may substitute for $L = 2W$ and $H = \dfrac{405}{LW} = \dfrac{405}{2WW} = \dfrac{405}{2W^2}$ to get

$C = 20W^2 + \dfrac{9 \cdot 405}{W}, \quad W > 0$

Then set $\dfrac{dC}{dW} = 40W - \dfrac{9 \cdot 405}{W^2} = 0$ and solve for W to get $W = 4.5$.

Since $L = 2W = 9$ and $H = \dfrac{405}{LW} = 10$, the value of $L + W + H$ is 23.5.

30. If three fair dice are thrown, and the sum is an odd number, what is the probability that all three dice show an odd number?

(a) 1/4 (b) 1/3 (c) 1/8 (d) 1/6 (e) 1/2

Solution:

The total number of ways of throwing an odd number on all 3 dice is $3^3 = 27$.

The only other way to have an odd sum is for exactly one of the other dice (3 ways) to be odd and the other two dice to be even. This can happen in $3 \times 3^3 = 81$ ways.

Therefore the probability that all dice are odd is $\dfrac{27}{27 + 81} = \dfrac{1}{4}$.