2004 STATE MATH CONTEST

SOLUTIONS – GRADES 10-12

1. The word WEBER hides a 5-digit number. Different letters indicate different digits, and same letters stand for the same digits. Every digit is a prime number and so is the sum of the 5 digits. The 2-digit number EW and the 3-digit number EBR are also primes. What digit does letter B represent?

   (a) 1 (b) 2 (c) 3 (d) 5 (e) 7

   Solution:

   The four different letters must stand for the four one-digit primes 2,3,5,7 in some order. Since the sum of these four numbers is 17, the only way the sum of the five numbers can be odd (necessary to make it prime) is if the duplicated number is 2. Therefore E stands for 2. Now W cannot be 5 or 7 (EW would not be prime), so W must be 3. Furthermore, R cannot be 5 (EBR would not be prime), so R=7, leaving B=5. Checking shows the numbers 3+2+5+2+7=19, 23 and 257 all to be primes, so B represents 5 and the correct answer is (d).

2. The last decimal digit of 2004^{2004} is

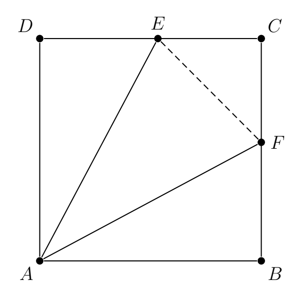
   (a) 0 (b) 2 (c) 4 (d) 6 (e) 8

   Solution:

   Notice that the last digit of the 2004^n alternates between 4 and 6, with the last digit being 4 for odd exponents, and 6 for even exponents. Since our exponent is even, the last digit will be 6, and the correct answer is (d).

3. In a square $ABCD$, points E and F on the boundary of the square are placed so that the line segments $\overline{AE}$ and $\overline{AF}$ divide the square into three parts of equal area. What is the ratio of the area of the triangle AEF to that of the square?

   (a) 3/18 (b) 4/18 (c) 5/18 (d) 6/18 (e) 7/18



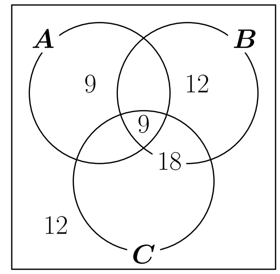
   Solution:

   For $\triangle ABF$ to have $1/3$ of the area of the square, BF must be $2/3$ of BC. Similarly, $DE = (2/3)DC$. Hence, the area of $\triangle ECF$ is $1/18$ of the area of the square. Since quadrilateral $AFCE$ covers $1/3$ of the square, and $1/18$ of the square is to be removed, $\triangle AEF$ covers $1/3 - 1/18 = 5/18$ of the square. The correct answer is (c).

4. In a survey of 69 people, only 9 liked all three of brands A, B and C; 12 didn't like any of the three; 9 liked only A; 30 disliked A but liked at least one of the other two. If 15 liked exactly two of the three, 12 liked only B, and 31 liked C, how many liked A and B but not C?

(a) 4 (b) 5 (c) 7 (d) 9 (e) none of these

Solution:



The easiest way to organize the information is in a Venn diagram. Out of the 30 who disliked A, but liked at least one of the other two, 12 liked B only, so there are 18 who liked either C only or B and C only.
Since 31 like C, and we already have 27 people in that set, there have to be exactly 4 who like A and C only. At this moment, all but one region has been counted. Given that there were 69 participants in the survey, the number of people who like A and B but not C must be $69 - (9 + 4 + 9 + 12 + 18 + 12) = 5$. Hence (b) is the correct answer.

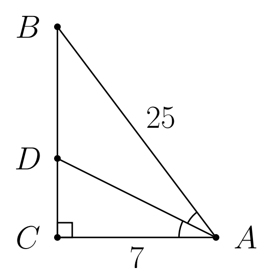
5. The product of the solutions to $4\left(2^{2x}\right) - 33\left(2^{x}\right) + 8 = 0$ is:

(a) 2 (b) $-3/4$ (c) -6 (d) 1 (e) none of these

Solution:

Introducing the new variable $y = 2^x$ simplifies our equation to $4y^2 - 33y + 8 = 0$. Solving this by either factoring or the quadratic formula yields $y = 8$ and $y = 1/4$ as the two solutions. (Yes, using the quadratic formula requires one to recognize that $\sqrt{961} = 31$, which in turn requires some number sense and faith in the solutions being rational numbers.) The corresponding values for x are 3 and -2, which multiply to -6. The correct answer is (c).

6. In the figure at the right, there is a right angle at C, and $\overline{AD}$ is a bisector of angle BAC. Given that $AC = 7$ and $AB = 25$, what is the length of segment CD?

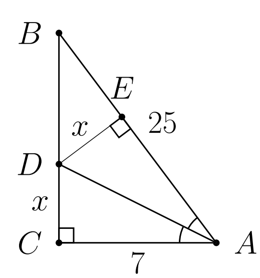


(a) $21/4$ (b) $24/7$ (c) 12 (d) $75/7$ (e) none of these

Solution:

I. One might know the theorem about the angle bisector dividing the opposite sides in proportion to the adjacent sides. If so, side BC, which has length 24 (Pythagorean theorem), must be divided by the angle bisector in the proportion 7:25. Therefore, $CD = 24 \times \dfrac{7}{32} = \dfrac{21}{4}$. The correct answer is (a).

II. Drawing a perpendicular from point D to side AB, and denoting the foot of the perpendicular by E yields $\triangle ADC \cong \triangle ADE$, since the triangles have one side and two angles congruent. Thus $DE = DC$. Denoting this length by x, the area of the big triangle, $\frac{7 \times 24}{2}$ must equal the sum of the areas of $\triangle ADC$ and $\triangle ADB$ or $\frac{7 \times x}{2} + \frac{25 \times x}{2}$. Solving the resulting equation yields $x = \frac{21}{4}$.



7. A single fair die is to be rolled until either a *one* or a *two* is rolled. Which of the following is most nearly equal to the probability that at least 3 rolls are needed?

(a) 0.3 (b) 0.4 (c) 0.5 (d) 0.6 (e) 0.7

Solution:

The probability that we will finish in one roll is $1/3$. The probability that we finish in exactly 2 rolls is $2/3 \times 1/3 = 2/9$. Since these are disjoint events, the probability that we finish in one or two rolls is $1/3 + 2/9 = 5/9$. Thus the probability that at least 3 rolls are needed is $1 - 5/9 = 4/9$. The closest number on the list is 0.4, (b) is the correct answer.

8. Tommy's house is next to a road that goes straight to Tommy's school. A railway runs parallel to that road. When Tommy leaves home at his regular time the moving train just catches him as he reaches school. One day Tommy left 4 minutes 10 seconds later and the train caught him one mile before he reached school. Given that Tommy always travels at 12 miles per hour, that the train always leaves on time, and the train always goes at the same constant speed, which of the following is the closest to the speed of the train?

(a) 50mph (b) 55mph (c) 60mph (d) 65mph (e) 70mph

Solution:

On the day when Tommy was late, and the train caught him earlier, he had 1 mile to bike to school from the meeting place. Tommy goes 1 mile in 5 minutes (given his speed of 12 mph). As he will be 4 minutes 10 seconds late from school the train passed him 50 seconds earlier than the usual time he and the train arrives at school. Therefore, it takes the train 50 seconds or $50/3600$ hours to travel 1 mile, so its speed is $1 \div (50/3600) = 72$ mph. The correct answer is (e).

9. How many 3-digit numbers contain three different digits, are between 300 and 800, and use only the digits 1,2,3,4,5,6,7,8,9?

(a) 280 (b) 336 (c) 360 (d) 405 (e) none of these

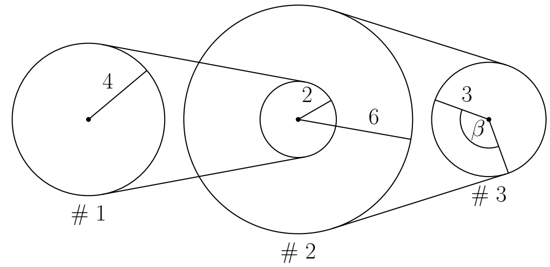
Solution:

The first digit must be one of 3, 4, 5, 6, or 7 to ensure that the resulting number falls between 300 and 800. After one of these 5 numbers is chosen, only 8 choices are left for the second digit, and 7 choices left for the third digit. Thus there are $5 \times 8 \times 7 = 280$ numbers satisfying the conditions of the problem, the correct answer is (a).

10. In the gear system below the two gears in the center (#2) turn together. The radius of gear #1 is 4 cm, the smaller radius in system #2 is 2 cm and the larger is 6 cm, and the radius of #3 is 3 cm. If gear #1 is turned through an angle $\theta = \frac{2\pi}{3}$, through what angle β will gear #3 be turned?

(a) 2π (b) $\dfrac{4\pi}{3}$ (c) $\dfrac{\pi}{6}$

(d) $\dfrac{8\pi}{3}$ (e) none of these



Solution:

The arc length corresponding to the angle $\theta = \frac{2\pi}{3}$ must be equal to the arc length of the corresponding revolution of the small gear #2. Since it has half the radius, twice the original angle, $\frac{4\pi}{3}$ will give the same arc length. When this revolution transfers to gear #3, the same thing happens, so the third gear will turn by angle $\beta = \frac{8\pi}{3}$. The correct answer is (d).

11. The notation $a\mathcal{M}b$ is defined to mean that a is a multiple of b and that a and b are both positive integers. Which of the following statements are always true for all positive integers w, x, y and z?

(i) $x\mathcal{M}y$ and $x\mathcal{M}z$ implies $x\mathcal{M}(yz)$

(ii) $x\mathcal{M}y$ and $x\mathcal{M}z$ implies $x\mathcal{M}(y+z)$

(iii) $x\mathcal{M}y$ and $y\mathcal{M}z$ implies $(x+y)\mathcal{M}z$

(iv) $x\mathcal{M}y$ and $y\mathcal{M}z$ implies $x\mathcal{M}z$

(v) $x\mathcal{M}y$ implies $y\mathcal{M}x$

(vi) $w\mathcal{M}x$ and $y\mathcal{M}z$ implies $(wy)\mathcal{M}(xz)$

(vii) $x\mathcal{M}y$ implies $(wx)\mathcal{M}(wy)$

(a) all of them (b) only (i), (ii), (iv), (vi) and (vii)

(c) only (iii), (iv), (vi) and (vii) (d) only (iv), (vi) and (vii) (e) none of these

Solution:

Let's look at every statement.

(i) Not true in general, for example $12\mathcal{M}6$ and $12\mathcal{M}4$, but $12\mathcal{M}24$ is not true.

(ii) We can use the same counterexample as above: $12\mathcal{M}6$ and $12\mathcal{M}4$, but $12\mathcal{M}10$ is not true.

(iii) This statement is true, if $x = ky$ and $y = lz$ for some integers k and l, then $x + y = ky + y = (k+1)y = (k+1)lz$ and thus $x + y$ is a multiple of z.

(iv) True again, just like before if $x = ky$ and $y = lz$ for some integers k and l, then $x = (kl)z$, a multiple of z.

(v) Not true in general, for example $6\mathcal{M}2$, but $2\mathcal{M}6$ is not true.

(vi) The statement is true. If $w = kx$ and $y = lz$, then $wy = (kl)xz$.

(vii) True again: if $x = ky$, then $wx = k(wy)$.

The true statements are (iii), (iv), (vi) and (vii); answer (c) is the correct one.

12. You are going for a vacation and have asked your neighbor to water your sick plant. The plant is in such a miserable condition that it will die even after watering with probability 0.2 and without watering with a probability of 0.8. You are 90% sure your neighbor will water the plant. What are the chances that the plant will die while you are on vacation?

(a) 16% (b) 26% (c) 50% (d) 74% (e) none of these

Solution:

From the Law of Total Probability (or just common sense), the probability that the plant will die is $90\% \times 0.2 + 10\% \times 0.8 = 26\%$. The correct answer is (b).

13. Which one of the following does not reduce to $\sin x$ for every x where the expressions are defined?

(a) $\dfrac{\tan x}{\sec x}$ (b) $\dfrac{\sin x}{\sec^2 x - \tan^2 x}$ (c) $\csc x - \cot x \cos x$

(d) $\dfrac{\sin^2 x \sec x}{\tan x}$ (e) all reduce to $\sin x$

Solution:

Standard identities provide one step reductions of all identities to $\sin x$ except for (c). There we need to make two steps: $\csc x - \cot x \cos x = \dfrac{1}{\sin x} - \dfrac{\cos x}{\sin x} \cos x = \dfrac{1 - \cos^2 x}{\sin x} = \dfrac{\sin^2 x}{\sin x} = \sin x$. Thus, all of the expressions reduce to $\sin x$ within their domain, the correct answer is (e).

14. The sum of the solutions to the equation $\sqrt{6x-2} - \sqrt{4x-3} = 1$ is:

(a) 4 (b) 3 (c) -3 (d) -4 (e) none of these

Solution:

The equation needs to be squared to remove the radicals. This operation might bring in incorrect solutions, so checking will be necessary at the end. To keep things simple, move $\sqrt{4x-3}$ to the right hand side before squaring. That yields $6x - 2 = 1 + 2\sqrt{4x-3} + 4x - 3$, which simplifies to $x = \sqrt{4x-3}$. Squaring this equation produces $x^2 = 4x - 3$. The quadratic equation can be solved by either factoring or using the quadratic formula to get two solutions $x = 1$ and $x = 3$. We must substitute in the original equation to check that both are correct solutions, their sum is 4, so the correct answer is (a).

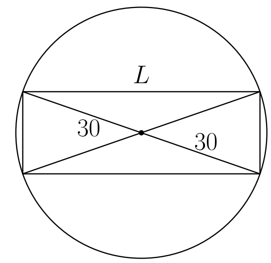
15. A rectangle is inscribed in a circle of radius 30. Which one of the following functions gives the area A of the rectangle in terms of the length L of the rectangle?

(a) $A = 60L - L^2$ (b) $A = L\sqrt{3600 - L^2}$ (c) $A = L\sqrt{900 - L^2}$

(d) $A = 900L - L^2$ (e) none of these

Solution:

Notice that the center of the circle must be where the diagonals of the rectangle intersect (it is the only point that is of equal distance from the four vertices). Using the Pythagorean Theorem, the width of the rectangle must be $\sqrt{(60)^2 - L^2}$, and so its area is $A = L\sqrt{3600 - L^2}$. The correct answer is (b).



16. Five friends Alex, Bill, Charlie, Daniel, and Eddy all own different brands of cars. Their cars are a Ford, a Buick, a Honda, an Audi and a Mercedes. One week they decided to try each others' cars and they traded cars every day to try all 5 cars. From Monday through Thursday none drove his own car and on Friday everybody drove his own car.

On Monday Daniel drove the Audi.

On Tuesday Charlie drove the Mercedes.

On Wednesday Charlie drove the Honda and Eddy the Mercedes.

On Thursday Alex had the Audi and Daniel had the Buick.

What kind of car does Bill have?

(a) Ford (b) Buick (c) Honda (d) Audi (e) Mercedes

Solution:

Let's start with Wednesday. The Honda and the Mercedes are used by Charlie and Eddie, Daniel must have driven one of the other three cars. He drove the Audi and Buick on other days, so Daniel must have driven the Ford on Wednesday. Since Alex drove the Audi on Thursday, that leaves Alex the Buick and Bill the Audi on Wednesday.

On Thursday, the Audi and the Buick are occupied, Charlie drove the Mercedes and the Honda on other days, he must have driven the Ford on Thursday. Who drove the Mercedes? Eddy didn't (he drove it on Wednesday), so Bill must have driven the Mercedes, and Eddy is left with the Honda.

What did Charlie drive on Monday? Not the Audi (Daniel drove it), and none of the Mercedes, Honda or Ford, since he drove them on other days. He must have driven the Buick on Monday. Similarly, Eddy must have driven the Ford, Bill the Honda and Alex the Mercedes.

Tuesday is now easy to figure out, Daniel drove the Honda, Alex the Ford, Bill the Buick and Eddy the Audi. Since Bill drove every car but the Ford during the first four days, he must have a Ford, the correct answer is (a).

One can organize the information into a 5x5 table and fill it out in one minute, following the above or some similar logic.

17. Solve $\begin{pmatrix} 1 & -2 \\ 4 & 3 \end{pmatrix} + 2\,\mathbf{X} = \begin{pmatrix} 2 & 1 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 4 \\ -3 & 1 \end{pmatrix}$ for $\mathbf{X}$.

(a) $\begin{pmatrix} 6 & 4 \\ -6 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 6 & 4 \\ -6 & 3 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 4 \\ -6 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} 7 & 2 \\ -2 & 5 \end{pmatrix}$ (e) none of these

Solution:

The first step can be to multiply the matrices on the right hand side of the equation to get

$$\begin{pmatrix} 1 & -2 \\ 4 & 3 \end{pmatrix} + 2\,\mathbf{X} = \begin{pmatrix} 13 & 6 \\ -8 & 7 \end{pmatrix},$$

$$2\,\mathbf{X} = \begin{pmatrix} 12 & 8 \\ -12 & 4 \end{pmatrix}, \quad \text{and} \quad \mathbf{X} = \begin{pmatrix} 6 & 4 \\ -6 & 2 \end{pmatrix}.$$

The correct answer is (a).

18. If $f(x) = \ln(6 - x)$ and $g(x) = |x^2 - 2x - 9|$ then what is the domain of $(f \circ g)(x) = f(g(x))$?

(a) $3 < x < 5$ (b) $3 \le x \le 5$ (c) $-3 < x < -1$ or $3 < x < 5$

(d) $-3 \le x \le -1$ or $3 \le x \le 5$ (e) none of these

Solution:

For $\ln(6-g(x))$ to be defined we need $g(x) < 6$, thus we have to solve the inequality $|x^2-2x-9| < 6$. This can be written as two inequalities:

$$-6 < x^2 - 2x - 9 < 6 \quad \text{and after completing the square} \quad 4 < (x-1)^2 < 16$$

from which either $-4 < x - 1 < -2$ or $2 < x - 1 < 4$. These yield $-3 < x < -1$ or $3 < x < 5$. The correct answer is (c).

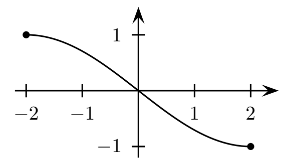
19. The center of a cube is reflected about every face of the cube, and the resulting points are connected to the nearest vertices of the cube. How many <u>edges</u> does the new solid have?

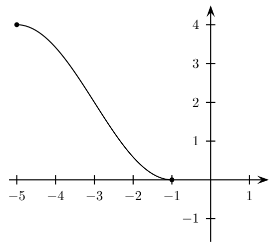
(a) 24 (b) 30 (c) 32 (d) 36 (e) 48

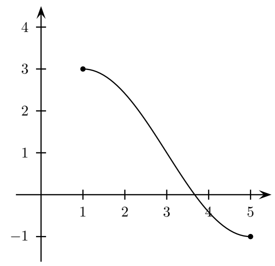
Solution:

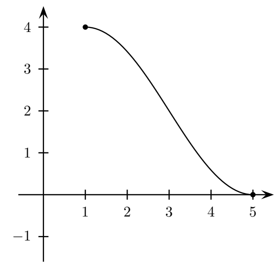
At first glance a square based pyramid is to be placed on every face of the cube, adding $6 \times 4 = 24$ additional edges to the existing 12. However, the original 12 edges of the cube will disappear. They are at the intersection of two triangular faces of adjacent pyramids, and those faces happen to be in the same plane, because of the reflections about the perpendicular faces. The solid has only 24 edges, the correct answer is (a).

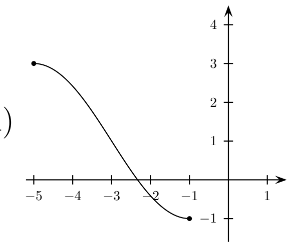
20. Given that the graph of $y = f(x)$ is a illustrated at the right, which one of the following is the graph of $y = 2f(x-3)+1$?



(a) 

(b) 

(c) 

(d) 

(e) none of these

Solution:

One can sketch (or imagine) the transformations necessary to change $f(x)$ into $2f(x-3)+1$. Shift to the right by 3 units, then stretch vertically by a factor of two (so that the range is from -2 to 2) and finally shift upward by 1 unit. The corresponding picture, and correct answer is (b).

21. If the surface area of one cube is 44% larger than the surface area of a second cube, which of the following is the closest to the percent that the volume of the first cube exceeds the volume of the second cube?

(a) 15 (b) 30 (c) 45 (d) 60 (e) 75

Solution:

I. Any two cubes are similar to each other. We can find the ratio of similarity (r) by knowing that the surface area changes by the square of this scaling factor. Thus $r^2 = 1.44$ and $r = 1.2$. The volume changes by the cube of the scaling factor. Since $r^3 = 1.728$, the closest answer to this 72.8% increase is the 75% increase of answer (e).

II. The second solution would require somewhat more calculation. Assume that the small cube has edges of length 1, and the large cube has edges of length x. Then using the surface areas, $1.44 \times 6 \times 1^2 = 6 \times x^2$, from which $x = 1.2$. The ratio of their volumes is $1.2^3/1^3 = 1.728$ which is closest to a 75% increase.

22. The natives on the island of Ooz form two tribes: the Veracities, who always tell the truth, and the Atrocities, who never tell the truth. Each has his name tattooed on his forehead, so names are not a problem and they are all well acquainted so that they know the true state of affairs. A traveller came upon a group of four natives and asked to which tribe each belonged. He received these answers:

Au: At least one of us is an Atrocity

Bo: Au is a Veracity

Co: We are not all Atrocities

Di: All of them are lying

To which tribe does each belong?

(a) All are Veracities (b) All are Atrocities (c) only Di is a Veracity

(d) All but Di are Veracities (e) Au and Bo are Veracities and the other two are Atrocities

Solution:

Au must be saying the truth, if he didn't, he would be an Atrocity making his own statement true. Thus Au is Veracity, and therefore Bo's and Co's sentences are also true, making them Veracities as well. Since the first three are telling the truth, Di is lying, making him the only Atrocity in the group. The correct answer is (d).

23. Consider the mean, median, mode and standard deviation of a set of $n \geq 3$ numbers, not all of them the same. If we leave out the smallest number (one of the smallest ones if two or more tie for smallest), the resulting $n-1$ numbers may have a different mean, median, mode and standard deviation. Which of the following statements MUST always be true:

(i) The mean of the numbers increases.

(ii) The median of the numbers increases.

(iii) The modes of the numbers remain the same.

(iv) The standard deviation of the numbers increases.

(a) none of them (b) only (i), (ii), and (iii)

(c) only (iii) and (iv) (d) only (i) and (ii) (e) only (i)

Solution:

(i) The mean of the numbers increases, since we left out the smallest one, and the numbers were not all equal.

(ii) The median may not change at all, like in the set 1,3,3,6 the median remains 3 after throwing out the 1 from the data.

(iii) The mode may change, for example the set 1,1,4,5,5,7 has two modes, 1 and 5, but after deleting the first 1, the only mode is 5.

(iv) The standard deviation actually may decrease, increase or remain the same after deleting the smallest value, as the following examples show:

$0,1,1 \quad \longrightarrow \qquad s = 1/\sqrt{3}, \quad s' = 0$

$0,0,1 \quad \longrightarrow \qquad s = 1/\sqrt{3}, \quad s' = 1/\sqrt{2}$

$0,0,1,1 \quad \longrightarrow \qquad s = 1/\sqrt{3}, \quad s' = 1/\sqrt{3}$

where s' is always obtained by calculating the standard deviation after deleting the smallest value from the set.

The only correct statement is (i), the correct answer is (e).

24. Given $i = \sqrt{-1}$, what is the value of the sum

$$\frac{1}{1+i} + \frac{1}{1-i} + \frac{1}{-1+i} + \frac{1}{-1-i} + \frac{2}{1+i} + \frac{2}{1-i} + \frac{2}{-1+i} + \frac{2}{-1-i} +$$

$$+ \frac{3}{1+i} + \frac{3}{1-i} + \frac{3}{-1+i} + \frac{3}{-1-i} + \ldots + \frac{n}{1+i} + \frac{n}{1-i} + \frac{n}{-1+i} + \frac{n}{-1-i}?$$

(a) $n^2 + n$ (b) $2n^2 + 2n$ (c) $2in^2 + 2in$ (d) $(1+i)n^2$ (e) none of these

Solution:

I. Notice that the first and fourth terms are opposites of each other, therefore add up to 0. The second and third terms are opposites too, their sum is also 0. The rest of the terms can be paired similarly, the $(4k+1)$st with the $(4k+4)$th and the $(4k+2)$nd with the $4k+3)$rd. Thus the entire sum is 0, the correct answer is (e).

II. Working with the first four terms only:

$$\frac{1}{1+i} + \frac{1}{1-i} + \frac{1}{-1+i} + \frac{1}{-1-i} = \frac{(1-i)+(1+i)}{(1+i)(1-i)} + \frac{(-1-i)+(-1+i)}{(-1+i)(-1-i)} = \frac{2}{2} - \frac{2}{2} = 0$$

The next four terms of the sum will also give 0, since they are twice the first four terms, and so on, the entire sum is 0. The correct answer is (e).

25. Write down all the integers from 1 to 30 to form the number

1234567891011121314....2930.

Now, delete 44 digits from this number and call the resulting number N. What is the possible value of N that is closest (smaller, larger or equal) to $5 \cdot 10^6$?

(a) 5,001,220 (b) 4,998,930 (c) 4,999,888 (d) 5,000,111 (e) none of these

Solution:

We have started with a 51 digit number. After deleting 44 of those digits we will have a 7 digit number left. We need to figure out which 44 digits to delete to get close to 5,000,000. It is easier to think of how to select 7 of the digits (without changing their order) to get close to 5,000,000. We do not have enough 0's to get exactly 5,000,000. If we try to get close to the number from below, we have to use a 4 as the first digit. Then we can use two 9's (one from 9, one from 19), but the last 9 from 29 cannot be used at the thousands place; there are not enough digits left after it. The largest number we can have less than 5,000,000 is therefore 4,998,930. On the other hand if we try to get close from above, we have to start with 5,0,0 but now a 1 must be used, and then 2's before the last digit of the sequence can be added. The smallest number greater than 5,000,000 is therefore 5,001,220. The closest to 5,000,000 is 4,998,930 the correct answer is (b).

26. Given that $f(x)$ is continuously differentiable on $a \leq x \leq b$ where $a < b$, $f(a) < 0$ and $f(b) > 0$, which of the following are always true?

(i) $f(x)$ is bounded on $a \leq x \leq b$

(ii) The equation $f(x) = 0$ has at least one solution in $a < x < b$

(iii) The maximum and minimum values of $f(x)$ on $a \leq x \leq b$ occur at points where $f'(c) = 0$

(iv) There is at least one point c with $a < c < b$ where $f'(c) > 0$

(v) There is at least one point d with $a < d < b$ where $f'(d) < 0$

(a) all true (b) only (ii) and (iv) are true (c) all but (iii) are true

(d) all but (v) are true (e) only (i), (ii) and (iv) are true

Solution:

(i) This statement is true, every continuous function is bounded on a closed interval.

(ii) True again, by the Intermediate Value Theorem.

(iii) Not true, because the maximum and/or minimum values could also occur at a or b, without the derivatives being 0.

(iv) True. By the Mean Value Theorem there exists a point between a and b where the derivative is exactly $\frac{f(b)-f(a)}{b-a}$, a clearly positive value.

(v) Not always true, for example the function might be strictly increasing guaranteeing the derivative to be always positive.

Thus the true statements are (i), (ii) and (iv), and the correct answer is (e).

27. For what values of x is $f(x) = \dfrac{1}{(x-1)(x+2)}$ increasing?

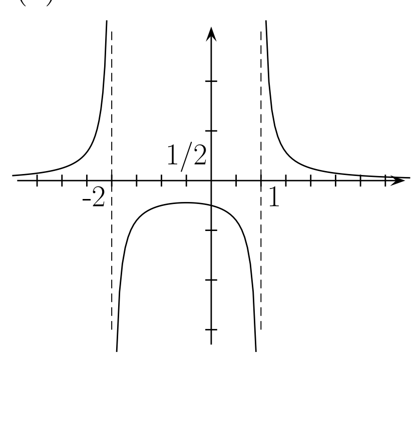
(a) $x > -\dfrac{1}{2}$, $x \neq 1$ (b) $x > -2$, $x \neq 1$ (c) $x < -1/2$

(d) $x < -2$ or $-2 < x < -1/2$ (e) none of these

Solution:

I. The function is undefined at 1 and -2, and differentiable everywhere else. It will be increasing where its derivative is positive. Easy calculation yields $f'(x) = \dfrac{-2x-1}{(x^2+x-2)^2}$. Since the denominator is always positive ($x \neq 1, -2$), we need $-2x - 1 > 0$, or $x < -1/2$. However, this interval contains -2, where the function is undefined, so the correct answer is (d).

II. One can solve this problem without using any calculus, by sketching the graph of $f(x)$. There are two vertical asymptotes at -2 and 1 and a horizontal asymptote of 0, at both $+\infty$ and $-\infty$. By checking the sign of the values near the vertical asymptotes on both sides, and observing the symmetry of the $\cap$ shape between -2 and 1, we can sketch the picture on the right, from which the intervals where the function is increasing are clearly those in answer (d).



28. The graph of $y = ax^2 + bx + 4$ passes through $(x, y) = (0, 4)$ for all values of a and b. Determine a and b such that the graph also passes through $(x, y) = (1, 3)$ and $(x, y) = (2, 6)$. The value of $a + b$ is:

(a) -7 (b) -6 (c) -1 (d) 1 (e) none of these

Solution:

I. By substituting the two given points into the equation we obtain: $3 = a \times 1^2 + b \times 1 + 4$ and $6 = a \times 2^2 + b \times 2 + 4$. Solving this system of equations with any standard technique yields $a = 2$, $b = -3$. The sum of these values is -1, the correct answer is (c).

II. A second (simpler) solution can be obtained by using only the first point $(1, 3)$. Substituting these coordinates into the equation of the parabola yields $3 = a + b + 4$, from which $a + b = -1$. We found the sum, without knowing either of the values of the two unknown parameters.

29. Let (s, t) be a point in the first quadrant (not on a coordinate axis) that is on the graph of $y = 9 - x^2$ and let $\mathcal{L}$ be a line tangent to $y = 9 - x^2$ at (s, t). Then $\mathcal{L}$ will cut off a triangle in the first quadrant. Find the (s, t) that corresponds to the triangle of that type that has the minimum area. What is the value of s?

(a)$\sqrt{3}$ (b) 2 (c)$\sqrt{2}$ (d) 3 (e) none of these

Solution:

The derivative of the function $9 - x^2$ is $-2x$, therefore the slope of the tangent line at point (s, t) is $-2s$. The equation of this tangent line is then $y - t = (x - s)(-2s)$. Since (s, t) is on the parabola, $t = 9 - s^2$, so the equation of the tangent line simplifies to $y = s^2 - 2sx + 9$. This line has a y-intercept of $s^2 + 9$ and x-intercept of $\frac{s^2+9}{2s}$. Thus the area of the triangle as a function of s is $A(s) = \frac{(s^2+9)^2}{4s}$. To minimize this function, we differentiate it,

$$A'(s) = \frac{(4s^3 + 36s)4s - 4(s^4 + 18s^2 + 81)}{16s^2} = \frac{12s^4 + 72s^2 - 324}{16s^2} = \frac{12(s^2 - 3)(s^2 + 9)}{16s^2}.$$

The minimum can occur at a value where the derivative is 0, and the only positive solution is at $s^2 = 3$, $s = \sqrt{3}$. It can be easily checked that this value is indeed a local minimum. The correct answer is (a).

30. The graph of function f contains the points $P(1, 2)$ and $Q(s, r)$. The equation of the secant line through P and Q is $y = \left(\dfrac{s^2 + 2s - 3}{s - 1}\right) x - 1 - s$. What is the value of $f'(1)$?

(a) 1 (b) 2 (c) 3 (d) 4 (e) none of these

Solution:

I. By definition, $f'(1)$ is the limit of the slope of the secant line when $s \longrightarrow 1$. Thus, $f'(1) = \lim_{s \to 1} \dfrac{s^2 + 2s - 3}{s - 1} = \lim_{s \to 1} \dfrac{(s - 1)(s + 3)}{s - 1} = \lim_{s \to 1}(s + 3) = 4$. The correct answer is (d).

II. We can get another solution by figuring out what f is. By substituting $x = s$ into the equation of the secant line, and cancelling by $s - 1$ again, we get $y = s^2 + 2s - 1$. This is $f(s)$, and its derivative is $f'(s) = 2s + 2$, so $f'(1) = 4$.