2004 STATE MATH CONTEST

GRADES 10 – 12

1. The word WEBER hides a 5-digit number. Different letters indicate different digits, and same letters stand for the same digits. Every digit is a prime number and so is the sum of the 5 digits. The 2-digit number EW and the 3-digit number EBR are also primes. What digit does letter B represent?

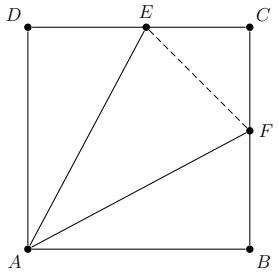
   (a) 1 (b) 2 (c) 3 (d) 5 (e) 7

2. The last decimal digit of 2004^{2004} is

   (a) 0 (b) 2 (c) 4 (d) 6 (e) 8

3. In a square $ABCD$, points E and F are placed on the boundary so that the line segments $\overline{AE}$ and $\overline{AF}$ divide the square into three parts of equal area. What is the ratio of the area of the triangle AEF to that of the square?

   (a) 3/18 (b) 4/18 (c) 5/18 (d) 6/18 (e) 7/18



4. In a survey of 69 people, only 9 liked all three of brands A, B and C; 12 didn't like any of the three; 9 liked only A; 30 disliked A but liked at least one of the other two. If 15 liked exactly two of the three, 12 liked only B, and 31 liked C, how many liked A and B but not C?

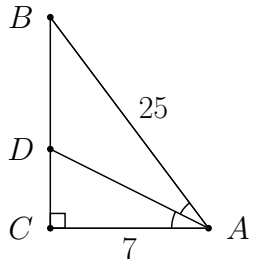
   (a) 4 (b) 5 (c) 7 (d) 9 (e) none of these

5. The product of the solutions to $4\left(2^{2x}\right) - 33\left(2^{x}\right) + 8 = 0$ is:

   (a) 2 (b) $-3/4$ (c) -6 (d) 1 (e) none of these

6. In the figure at the right, there is a right angle at C, and $\overline{AD}$ is a bisector of angle BAC. Given that $AC = 7$ and $AB = 25$, what is the length of segment CD?

   (a) 21/4 (b) 24/7 (c) 12 (d) 75/7 (e) none of these



7. A single fair die is to be rolled until either a *one* or a *two* is rolled. Which of the following is most nearly equal to the probability that at least 3 rolls are needed?

   (a) 0.3 (b) 0.4 (c) 0.5 (d) 0.6 (e) 0.7

8. Tommy's house is next to a road that goes straight to Tommy's school. A railway runs parallel to that road. When Tommy leaves home at his regular time the moving train just catches him as he reaches school. One day Tommy left 4 minutes 10 seconds later and the train caught him one mile before he reached school. Given that Tommy always travels at 12 miles per hour, that the train always leaves on time, and the train always bikes at the same constant speed, which of the following is the closest to the speed of the train?

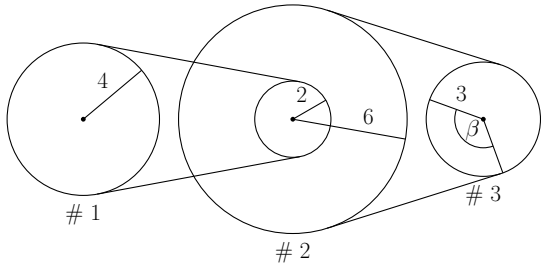
   (a) 50mph (b) 55mph (c) 60mph (d) 65mph (e) 70mph

9. How many 3-digit numbers contain three different digits, are between 300 and 800, and use only the digits 1,2,3,4,5,6,7,8,9?

   (a) 280 (b) 336 (c) 360 (d) 405 (e) none of these

10. In the gear system below the two gears in the center (#2) turn together. The radius of gear #1 is 4 cm, the smaller radius in system #2 is 2 cm and the larger is 6 cm, and the radius of #3 is 3 cm. If gear #1 is turned through an angle $\theta = \frac{2\pi}{3}$, through what angle β will gear #3 be turned?

    (a) 2π (b) $\frac{4\pi}{3}$ (c) $\frac{\pi}{6}$
    (d) $\frac{8\pi}{3}$ (e) none of these



11. The notation $a\mathcal{M}b$ is defined to mean that a is a multiple of b and that a and b are both positive integers. Which of the following statements are always true for all positive integers w, x, y and z?

    (i) $x\mathcal{M}y$ and $x\mathcal{M}z$ implies $x\mathcal{M}(yz)$
    (ii) $x\mathcal{M}y$ and $x\mathcal{M}z$ implies $x\mathcal{M}(y+z)$
    (iii) $x\mathcal{M}y$ and $y\mathcal{M}z$ implies $(x+y)\mathcal{M}z$
    (iv) $x\mathcal{M}y$ and $y\mathcal{M}z$ implies $x\mathcal{M}z$
    (v) $x\mathcal{M}y$ implies $y\mathcal{M}x$
    (vi) $w\mathcal{M}x$ and $y\mathcal{M}z$ implies $(wy)\mathcal{M}(xz)$
    (vii) $x\mathcal{M}y$ implies $(wx)\mathcal{M}(wy)$

    (a) all of them (b) only (i), (ii), (iv), (vi) and (vii)
    (c) only (iii), (iv), (vi) and (vii) (d) only (iv), (vi) and (vii) (e) none of these

12. You are going for a vacation and have asked your neighbor to water your sick plant. The plant is in such a miserable condition that it will die even after watering with probability 0.2 and without watering with a probability of 0.8. You are 90% sure your neighbor will water the plant. What are the chances that the plant will die while you are on vacation?

(a) 16% (b) 26% (c) 50% (d) 74% (e) none of these

13. Which one of the following does not reduce to $\sin x$ for every x where the expressions are defined?

(a) $\dfrac{\tan x}{\sec x}$ (b) $\dfrac{\sin x}{\sec^2 x - \tan^2 x}$ (c) $\csc x - \cot x \cos x$

(d) $\dfrac{\sin^2 x \sec x}{\tan x}$ (e) all reduce to $\sin x$

14. The sum of the solutions to the equation $\sqrt{6x-2} - \sqrt{4x-3} = 1$ is:

(a) 4 (b) 3 (c) -3 (d) -4 (e) none of these

15. A rectangle is inscribed in a circle of radius 30. Which one of the following functions gives the area A of the rectangle in terms of the length L of the rectangle?

(a) $A = 60L - L^2$ (b) $A = L\sqrt{3600 - L^2}$ (c) $A = L\sqrt{900 - L^2}$

(d) $A = 900L - L^2$ (e) none of these

16. Five friends Alex, Bill, Charlie, Daniel, and Eddy all own different brands of cars. Their cars are a Ford, a Buick, a Honda, an Audi and a Mercedes. One week they decided to try each others' cars and they traded cars every day to try all 5 cars. From Monday through Thursday none drove his own car and on Friday everybody drove his own car.

On Monday Daniel drove the Audi.

On Tuesday Charlie drove the Mercedes.

On Wednesday Charlie drove the Honda and Eddy the Mercedes.

On Thursday Alex had the Audi and Daniel had the Buick.

What kind of car does Bill have?

(a) Ford (b) Buick (c) Honda (d) Audi (e) Mercedes

17. Solve $\begin{pmatrix} 1 & -2 \\ 4 & 3 \end{pmatrix} + 2\,\mathbf{X} = \begin{pmatrix} 2 & 1 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 4 \\ -3 & 1 \end{pmatrix}$ for $\mathbf{X}$.

(a) $\begin{pmatrix} 6 & 4 \\ -6 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 6 & 4 \\ -6 & 3 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 4 \\ -6 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} 7 & 2 \\ -2 & 5 \end{pmatrix}$ (e) none of these

18. If $f(x) = \ln(6 - x)$ and $g(x) = |x^2 - 2x - 9|$ then what is the domain of $(f \circ g)(x) = f(g(x))$?

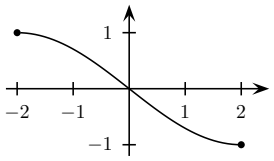
(a) $3 < x < 5$ (b) $3 \leq x \leq 5$ (c) $-3 < x < -1$ or $3 < x < 5$

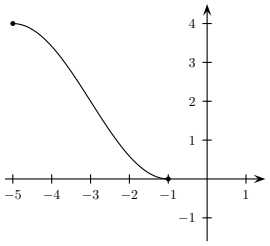
(d) $-3 \leq x \leq -1$ or $3 \leq x \leq 5$ (e) none of these

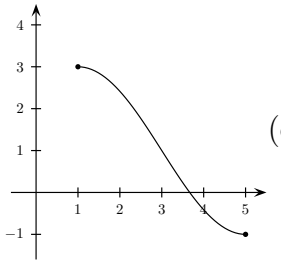
19. The center of a cube is reflected about every face of the cube, and the resulting points are connected to the nearest vertices of the cube. How many edges does the new solid have?

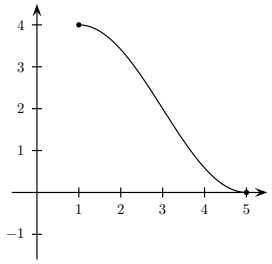
(a) 24 (b) 30 (c) 32 (d) 36 (e) 48

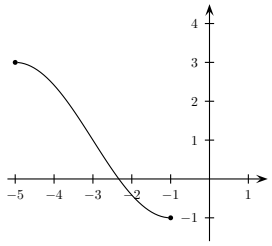
20. Given that the graph of $y = f(x)$ is a illustrated at the right, which one of the following is the graph of $y = 2f(x - 3) + 1$?



(a) 

(b) 

(c) 

(d) 

(e) none of these

21. If the surface area of one cube is 44% larger than the surface area of a second cube, which of the following is the closest to the percent that the volume of the first cube exceeds the volume of the second cube?

(a) 15 (b) 30 (c) 45 (d) 60 (e) 75

22. The natives on the island of Ooz form two tribes: the Veracities, who always tell the truth, and the Atrocities, who never tell the truth. Each has his name tattooed on his forehead, so names are not a problem and they are all well acquainted so that they know the true state of affairs. A traveller came upon a group of four natives and asked to which tribe each belonged. He received these answers:

Au: At least one of us is an Atrocity

Bo: Au is a Veracity

Co: We are not all Atrocities

Di: All of them are lying

To which tribe does each belong?

(a) All are Veracities (b) All are Atrocities (c) only Di is a Veracity

(d) All but Di are Veracities (e) Au and Bo are Veracities and the other two are Atrocities

23. Consider the mean, median, mode and standard deviation of a set of $n \geq 3$ numbers, not all of them the same. If we leave out the smallest number (one of the smallest ones if two or more tie for smallest), the resulting $n-1$ numbers may have a different mean, median, mode and standard deviation. Which of the following statements MUST always be true:

(i) The mean of the numbers increases.

(ii) The median of the numbers increases.

(iii) The modes of the numbers remain the same.

(iv) The standard deviation of the numbers increases.

(a) none of them (b) only (i), (ii), and (iii)

(c) only (iii) and (iv) (d) only (i) and (ii) (e) only (i)

24. Given $i = \sqrt{-1}$, what is the value of the sum

$$\frac{1}{1+i} + \frac{1}{1-i} + \frac{1}{-1+i} + \frac{1}{-1-i} + \frac{2}{1+i} + \frac{2}{1-i} + \frac{2}{-1+i} + \frac{2}{-1-i} +$$

$$+\frac{3}{1+i} + \frac{3}{1-i} + \frac{3}{-1+i} + \frac{3}{-1-i} + \ldots + \frac{n}{1+i} + \frac{n}{1-i} + \frac{n}{-1+i} + \frac{n}{-1-i}?$$

(a) $n^2 + n$ (b) $2n^2 + 2n$ (c) $2in^2 + 2in$ (d) $(1+i)n^2$ (e) none of these

25. Write down all the integers from 1 to 30 to form the number

1234567891011121314....2930.

Now, delete 44 digits from this number and call the resulting number N. What is the possible value of N that is closest (smaller, larger or equal) to $5 \cdot 10^6$?

(a) 5,001,220 (b) 4,998,930 (c) 4,999,888 (d) 5,000,111 (e) none of these

26. Given that $f(x)$ is continuously differentiable on $a \leq x \leq b$ where $a < b$, $f(a) < 0$ and $f(b) > 0$, which of the following are always true?

(i) $f(x)$ is bounded on $a \leq x \leq b$

(ii) The equation $f(x) = 0$ has at least one solution in $a < x < b$

(iii) The maximum and minimum values of $f(x)$ on $a \leq x \leq b$ occur at points where $f'(c) = 0$

(iv) There is at least one point c with $a < c < b$ where $f'(c) > 0$

(v) There is at least one point d with $a < d < b$ where $f'(d) < 0$

(a) all true (b) only (ii) and (iv) are true (c) all but (iii) are true

(d) all but (v) are true (e) only (i), (ii) and (iv) are true

27. For what values of x is $f(x) = \dfrac{1}{(x-1)(x+2)}$ increasing?

(a) $x > -\dfrac{1}{2}$, $x \neq 1$ (b) $x > -2$, $x \neq 1$ (c) $x < -1/2$

(d) $x < -2$ or $-2 < x < -1/2$ (e) none of these

28. The graph of $y = ax^2+bx+4$ passes through $(x, y) = (0, 4)$ for all values of a and b. Determine a and b such that the graph also passes through $(x, y) = (1, 3)$ and $(x, y) = (2, 6)$. The value of $a + b$ is:

(a) -7 (b) -6 (c) -1 (d) 1 (e) none of these

29. Let (s, t) be a point in the first quadrant (not on a coordinate axis) that is on the graph of $y = 9 - x^2$ and let $\mathcal{L}$ be a line tangent to $y = 9 - x^2$ at (s, t). Then $\mathcal{L}$ will cut off a triangle in the first quadrant. Find the (s, t) that corresponds to the triangle of that type that has the minimum area. What is s?

(a)$\sqrt{3}$ (b) 2 (c)$\sqrt{2}$ (d) 3 (e) none of these

30. The graph of function f contains the points $P(1, 2)$ and $Q(s, r)$. The equation of the secant line through P and Q is $y = \left(\dfrac{s^2 + 2s - 3}{s - 1}\right) x - 1 - s$. What is the value of $f'(1)$?

(a) 1 (b) 2 (c) 3 (d) 4 (e) none of these