2004 STATE MATH CONTEST

JUNIOR PRETEST – SOLUTIONS

1. Find the value of the alternating sum $1-2+3-4+\ldots+2003-2004$.

   (a) -501 (b) -1002 (c) -2003 (d) - 2505 (e) 1002

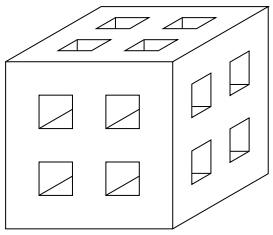
   Solution. In finite sums, any grouping of terms is admissible. Thus we have

   $$1-2+3-4+\ldots+2003-2004=(1-2)+(3-4)+(5-6)+\ldots+(2001-2002)+(2003-2004)=$$
   $$=\underbrace{(-1)+(-1)+\ldots+(-1)}_{2004/2 \text{ parenthesis}}=1002\cdot(-1)=-1002.$$

   The correct answer is (b).

2. In a 5 by 5 by 5 cube, four symmetrically placed 1 by 1 square holes are cut perpendicular to each face all the way through to the opposite face. (See the picture at the right.) How many small 1 by 1 by 1 cubes are left after this construction?

   (a) 81 (b) 61 (c) 75 (d) 105 (e) 96

   Solution. Before cutting any holes there are $5^3=125$ small cubes in the 5 by 5 by 5 cube. When four holes are first cut through two opposite faces, say the front and back faces, $4\cdot5=20$ small (1 by 1 by 1) cubes are removed. Cutting through the left and right faces removes an additional $4\cdot(5-2)=12$ small cubes, as does cutting through the top and bottom faces. This leaves $125-20-12-12=81$ small cubes. The correct answer is (a).

3. If a certain number is both subtracted from the numerator and added to the denominator of the fraction $\frac{29}{31}$, the resulting fraction simplifies to $\frac{1}{7}$. The number is closest to:

   (a) 20.3 (b) 21.3 (c) 22.3 (d) 23.3 (e)24.3

   Solution. Using x for the unknown number, the resulting equation is $\frac{29-x}{31+x}=\frac{1}{7}$. Cross-multiplying yields $7(29-x)=(31+x)$ and therefore $x=\frac{172}{8}=21.5$. The correct answer is (b).

4. Five years ago a girl was one fifth the age of her mother. In twenty two years she will be half as old as her mother. Find the sum of the ages of the girl and her mother.

(a) 42 yrs (b) 46 yrs (c) 50 yrs (d) 54 yrs (e) none of these

Solution. Using g as the girl's present age and m as the mother's present age, the conditions yield the following system of equations:

$$\begin{cases} (g-5) &= \frac{1}{5}(m-5) \\ (g+22) &= \frac{1}{2}(m+22) \end{cases}$$

To eliminate m, multiply the first equation by 5, the second by -2 and add. The result is $g = 14$ and $m = 50$. The sum of ages of the girl and her mother is 64. The correct answer is (e).

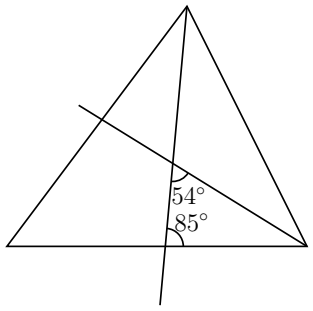
Check: Five years ago, the girl was 9 and her mother was 5 times older (she was 45). In 22 years, the girl will be 36 and her mother 72, so the mother will be twice as old as the daughter.

5. A sphere is circumscribed around a rectangular box having dimensions 4 by 12 by 18. What is the radius of this sphere?

(a) 10 (b) 11 (c) 12 (d) 13 (e) 14

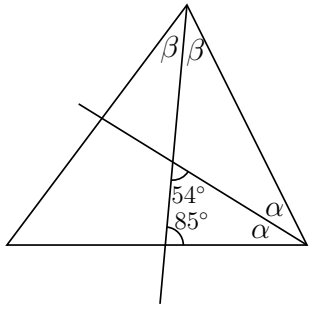
By symmetry, the sphere will pass through each corner of the box and the center of the sphere will be the center of the box. Hence, the radius of the sphere is the distance from the center of the box to any corner. To measure this distance, first move $\dfrac{4}{2} = 2$ units from the center to the smallest side, then $\dfrac{18}{2} = 9$ units parallel to the longest side, and then $\dfrac{12}{2} = 6$ units parallel to the other side. By the Pythagorean Theorem this distance is $R = \sqrt{2^2 + 6^2 + 9^2} = \sqrt{121} = 11$. The correct answer is (b).

6. An angle bisector of a triangle forms a angle of 85° with the opposite side and a angle of 54° with another angle bisector, as pictured. How large is the smallest angle of the triangle?



(a) 20° (b) 26° (c) 32° (d) 34° (e) 40°

Solution. Let α and β be as on the diagram. Then $\alpha = 180^\circ - (85^\circ + 54^\circ) = 41^\circ$ and $\beta = 180^\circ - (126^\circ + 41^\circ) = 13^\circ$. So one angle of the triangle is $2 \times 41^\circ = 82^\circ$, a second angle is $2 \times 13 = 26^\circ$; and the third is $180^\circ - 82^\circ - 26^\circ = 72^\circ$. Thus the smallest angle is 26°. The correct answer is (b).



7. One barrel is full of oil, and an identical barrel is half full. The full barrel weighs 86kg, and the half full barrel weighs 53kg. What is the weight of an empty barrel?

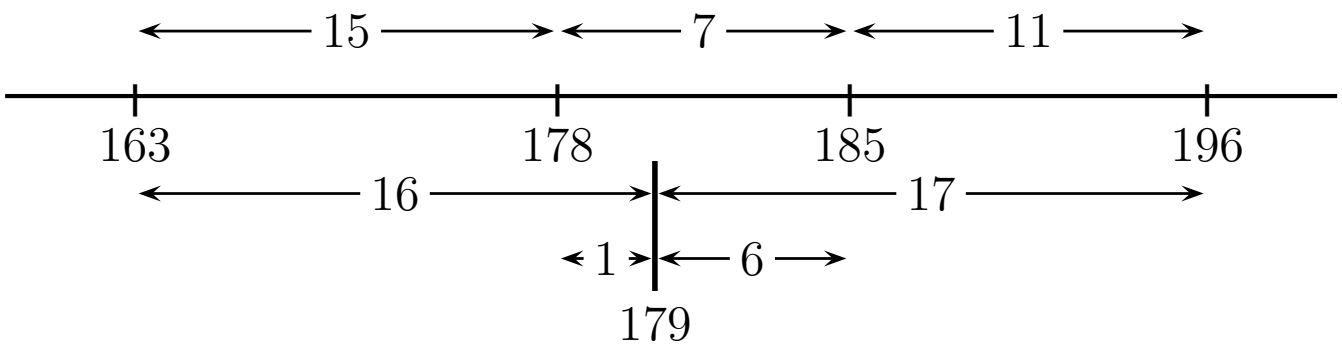
(a) 12 (b) 14 (c) 16 (d) 18 (e) 20

Solution. The difference of the given weights represents weight of the oil that fills half of the barrel. Thus the amount of oil in the half full barrel is $86 - 53 = 33$kg, so the weight of the barrel is $53 - 33 = 20$kg. The correct answer is (e).

Check. $20+33 = 53$kg gives the weight of the barrel that is half filled with oil and $20+2\cdot 33 = 86$kg gives the weight of the other barrel.

8. Four children guess the height of their teacher to be 196cm, 163cm, 178cm, and 185cm. They all missed the exact height and they were off (in some order) by 1 cm, 6cm, 16cm and 17cm. What is the sum of the digits in the teacher's real height?

(a) 15 (b) 16 (c) 17 (d) 18 (e) 19



Solution. Let H be true height of the teacher. If H is 1cm from 163cm, then H is either 162 or 164. But then the estimate of 196cm is too large (the error is bigger than 17cm). Thus H cannot be 1cm from 163. In a similar fashion it cannot be 1cm from 196. Thus H is either 177, 179, 184 or 186.

If H is 177, then $H - 163 = 14$, which is not possible. Similarly if H is 184 or 186.

$H = 179$cm is the true height. The sum of the digits is 17. The correct answer is (c).

9. Suppose that all three digit numbers (100 through 999) were written individually on pieces of paper and thrown in a hat. At least how many pieces of paper have to be randomly selected, so that without looking at these, one can be certain that at least two of the pieces of paper selected will have numbers whose digits sum to the same thing?

(a) 10 (b) 18 (c) 28 (d) 40 (e) 50

Solution. The smallest possible digit sum is equal to 1 (for the number 100) and the largest possible sum is equal to 27 (for the number 999). Clearly, any number between 1 and 27 can also be obtained as a digit sum. So in the worst-possible scenario, we can at first select 27 pieces of papers with different digit sums, but then on the 28th piece of paper must have a digit sum equal to one of the others. Thus $n = 28$. The correct answer is (c).

10. The symbol + is for addition and $\times$ is for multiplication. If one were to leave the "+" and "$\times$" signs where they are in $2 \times 3 + 5$ (which total 11) but interchange digits and/or insert grouping symbols as in $2 \times (3 + 5)$ (which totals 16) and $3 \times 5 + 2$ (which totals 17), how many different values would be possible?

(a) 4 (b) 6 (c) 8 (d) 10 (e) none of these

Solution. Since both multiplication and addition are commutative, the following is a list all possibilities:

$$\begin{aligned} 2 \times 3 + 5 &= 11 \\ 2 \times 5 + 3 &= 13 \\ 3 \times 5 + 2 &= 17 \\ 2 \times (3 + 5) &= 16 \\ 3 \times (2 + 5) &= 21 \\ 5 \times (2 + 3) &= 25 \end{aligned}$$

The correct answer is (b).

11. The six digit number $15a64b$ is divisible by 45. Which one of the following could (not must) be the product of the two missing digits a and b?

(a) 12 (b) 30 (c) 2 (d) 20 (e) 10

Solution. As $45 = 5 \cdot 9$, the divisibility by 5 implies that b is either 0 or 5. If b is zero, $a \times b$ is zero, and that is not on the list, so let's explore the possibility that $b = 5$. In that case the sum of digits becomes $1 + 5 + a + 6 + 4 + 5 = 21 + a$. Thus the divisibility by 9 implies that $21 + a$ is divisible by 9. Since $0 \leq a \leq 9$, it must be the case that $21 + a = 27$, making $a = 6$. For $a = 6$ and $b = 5$, $a \times b$ is 30, which is listed. Thus the correct answer is (b).

12. The gas mileage of a car is

16 miles per gallon at 80 miles per hour,

18 miles per gallon at 70 miles per hour,

20 miles per gallon at 60 miles per hour.

If that car were driven at 80 mph for 2 hours, at 70 mph for 3 hours, and at 60 mph for 5 hours, about how many gallons of gas would be used?

(a) 35 (b) 37 (c) 39 (d) 41 (e) 43

Solution. At a speed of 80 mph, the car uses $\dfrac{80}{16} = 5$ gallons during each hour.

At a speed of 70 mph, the car uses $\dfrac{70}{18}$ gallons during each hour.

At a speed of 60 mph, the car uses $\dfrac{60}{20} = 3$ gallons during each hour.

Thus the number of gallons used was $2 \times \dfrac{80}{16} + 3 \times \dfrac{70}{18} + 5 \times \dfrac{60}{20} = 36\dfrac{2}{3}$. The correct answer is (b).

13. The graph of $y = ax^2+bx+4$ passes through $(x, y) = (0, 4)$ for all values of a and b. Determine a and b such that the graph also passes through $(x, y) = (1, 3)$ and $(x, y) = (2, 6)$. The value of $a + b$ is:

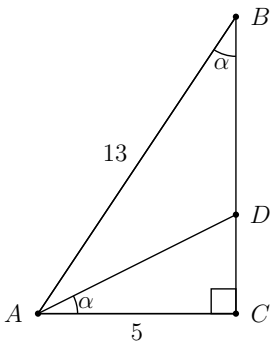
(a)1 (b)2 (c) 0 (d)-2 (e) -1

Solution. If a point (x_0, y_0) lies on the graph of $y = ax^2 + bx + c$, the coordinates of the point must satisfy the function's equation: $y_0 = ax_0^2 + bx_0 + 4$. Using the points $(x_0, y_0) = (1, 3)$ and $(x_0, y_0) = (2, 6)$ yields the following system of equations:

$$\begin{cases} a \cdot 1^2 + b \cdot 1 + 4 &= 3 \\ a \cdot 2^2 + b \cdot 2 + 4 &= 6 \end{cases}$$

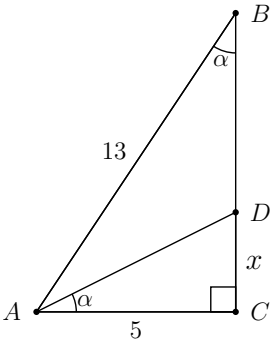
The system has solutions $a = 2,\ b = -3$, so $a + b = -1$. The correct answer is (e).

14. In $\triangle ACB$ at the right, $\angle C = 90°$, $\overline{AC} = 5$ and $\overline{AB} = 13$. A line is drawn from A to D on $\overline{BC}$ making an angle of measure α with $\overline{AC}$, which just happens to be the measure of $\angle B$. Compute the ratio of $\overline{BD}$ to $\overline{CD}$.



(a) $\frac{119}{25}$ (b) 4 (c) $\frac{144}{25}$ (d) 5 (e) $\frac{169}{25}$

Solution. By the Pythagorean theorem $\overline{BC} = \sqrt{13^2 - 5^2} = 12$. So if we let $x = \overline{CD}$, then $\overline{BD} = 12 - x$.
Triangles $\triangle ACD$ and $\triangle BCA$ are similar because they have two corresponding equal angles, so



$$\frac{12}{5} = \frac{5}{x}.$$

Thus $x = \frac{25}{12}$, so $\overline{CD} = \frac{25}{12}$ and $\overline{BD} = 12 - \frac{25}{12} = \frac{119}{25}$. Therefore $\frac{\overline{BD}}{\overline{CD}} = \frac{119}{25}$. The correct answer is (a).

15. Which of the following are always true for all nonzero real numbers x, y and z.

i. $(x \cdot y) \div z = x \cdot (y \div z)$

ii. $(x \div y) \div z = x \div (y \div z)$

iii. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

iv. $(x \div y) \cdot z = x \div (y \div z)$

v. $x \div (y \cdot z) = x \div (y \div z)$

(a) only i, iii (b) only i, iv (c) only ii, iii, iv (d) only i, iii, iv (e) all of the above

Solution.

i. TRUE. $(x \cdot y) \div z = \dfrac{xy}{z} = x \cdot \dfrac{y}{z} = x \cdot (y \div z)$

ii. FALSE. $(x \div y) \div z = \dfrac{\left(\frac{x}{y}\right)}{z} = \dfrac{x}{yz} \neq \dfrac{xz}{y} = \dfrac{x}{\left(\frac{y}{z}\right)} = x \div (y \div z)$

iii. TRUE. Multiplication is associative so $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

iv. TRUE. $(x \div y) \cdot z = \dfrac{x}{y} \cdot z = \dfrac{xz}{y} = \dfrac{x}{\left(\frac{y}{z}\right)} = x \div (y \div z)$

v. FALSE. $x \div (y \cdot z) = \dfrac{x}{yz} \neq \dfrac{xz}{y} = \dfrac{x}{\left(\frac{y}{z}\right)} = x \div (y \div z)$

The correct answer is (d).