SOLUTIONS
2004 SENIOR PRETEST

1. How many natural number divisors exist for $3^4 \cdot 7 \cdot 11^2$?

   (a) 8 (b) 10 (c) 30 (d) 32 (e) none of these

   ANSWER: (c)

   Method I. One can list every divisor; there are 30 of these, but organizing the list, so that none will be left out or counted twice, is difficult.

   Method II. The quick way to solve the problem is to realize that a divisor will contain either 0, 1, 2, 3 or 4 factors of 3, a total of 5 choices; either 0 or 1 factors of 7; and either 0, 1, or 2 factors of 11. The total number of choices to make for a divisor is $5 \times 2 \times 3 = 30$, since choosing the different factors are independent. Any such choice results in a different divisor, and every divisor can be obtained this way. The number has 30 divisors.

2. For all positive numbers x and y, the operation $*$ is defined to be $x * y = y^x$. Which of the following choices are always true for all positive numbers x, y and z?

   (i) $1 * y = y$ (ii) $x * x = x^2$ (iii) $x * y = y * x$

   (iv) $(x * y) * z = x * (y * z)$ (v) $(x + y) * z = x * z + y * z$

   (a) all are true (b) none are true (c) only (i) is true

   (d) only (i) and (iv) are true (e) only (i), (iv) and (v) are true

   ANSWER: (c)

   The statements are easy to examine if we rewrite them in standard notation:

   (i) $1 * y = y$ translates to $y^1 = y$, which is clearly true.

   (ii) $x^x = x^2$ is true in a few special cases, but not true in general. Try $x = 3$.

   (iii) $x^y = y^x$ is often not true all the time, for example $2^3 = 8 \neq 3^2 = 9$.

   (iv) $z^{(y^x)}$ and $(z^y)^x$ look very similar. However, substituting some numbers usually gives different results. For example $2^{(2^3)} = 2^8$ and $(2^2)^3 = 2^6$.

   (v) $z^{(x+y)} = z^x + z^y$, is false. For instance, $1^{(2+3)} = 1$ but $1^2 + 1^3 = 2$.

   Only statement (i) is true for all positive numbers x and y.

3. The first three terms of an arithmetic sequence, in order, are $2x+4$, $5x-4$ and $3x+4$. What is the sum of the first 10 terms of this sequence?

   (a) 176 (b) 202.4 (c) 352 (d) 396 (e) none of these

   ANSWER: (a)

   The difference between the first two terms must be the same as the difference between the second and third terms. This yields $3x - 8 = -2x + 8$ which gives $x = 16/5$. One could

obtain the sum by computing all ten terms and adding them, but easier is to realize that the the sum of the first and last terms gives the same result as the sum of the second and next to last, etc. This leads to the well known formula for the sum for the sum of an arithmetic sequence. Using $x = 16/5$, the first term of the sequence is 10.4, and the common difference of the sequence is 1.6. The 10th term then becomes $10.4 + 9 \times 1.6 = 24.8$, so the sum of the first 10 terms is $S = (10.4 + 24.8) \times 10/2 = 176$.

4. The line $y = \frac{1}{2}x + B$ is tangent to the graph of $y = -\frac{1}{4}(x^2 - 16x + 49)$. What is the value of B?

(a) $-\frac{1}{3}$ (b) 0 (c) $\frac{1}{3}$ (d) $\frac{3}{4}$ (e) $\frac{4}{3}$

ANSWER: (b)

Method I. The first solution uses some easy calculus. The slope of the tangent line is 1/2, so we can find the point of tangency P by differentiating the function:

$$\frac{dy}{dx} = -\frac{1}{2}x + 4 = 1/2$$

Solving this yields $x = 7$; the corresponding y-value is $y = 7/2$ from the quadratic equation. We must fit the line $y = (1/2)x + B$ to the point $P(7, 7/2)$. Plugging in the numbers yields $B = 0$.

Method II. Here is a "calculus free" solution next. For the line to be tangent, the parabola and the line must have exactly one common point. Thus we have to find a value of B for which the resulting equation system has a unique solution. This equation system reduces to the quadratic equation

$$\frac{1}{2}x + B = -\frac{1}{4}(x^2 - 16x + 49),$$

which further simplifies to $x^2 - 14x + 49 + 4B$. To have only one solution the discriminant of this equation must be 0, i.e. $196 - 4(49 + 4B) = 0$, from which we get $B = 0$. A sketch of a picture can convince us that this unique solution must indeed give a tangent line.

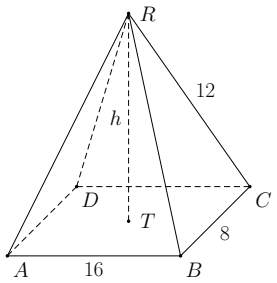
5. If three fair dice are rolled, what is the probability that none will show six and all will show different numbers?

(a) $\frac{5}{18}$ (b) $\frac{125}{216}$ (c) $\frac{5}{9}$ (d) $\frac{1}{8}$ (e) none of these

ANSWER: (a)

Think of the three dice as being colored differently so we can easily distinguish between them, then $6 \times 6 \times 6$ equally likely outcomes exist for the roll. How many of these are favorable? The first die can show 5 numbers (anything but 6), the second die thus has 4 possibilities, and the third only 3. Thus, the probability of a successful roll is $\frac{5 \times 4 \times 3}{6 \times 6 \times 6} = \frac{5}{18}$.

6. The pyramid at the right has an 8 by 16 rectangular base; its other four edges are equal to 12. Which of the following is closest in value to the height h?



(a) 7.5 (b) 8 (c) 8.5 (d) 9 (e) 9.5

ANSWER: (b)

We need to use the Pythagorean theorem twice. Triangle ABC is a right triangle, so $AC = \sqrt{16^2 + 8^2} = \sqrt{320}$, so $TC = \frac{1}{2}AC = \sqrt{80}$. In right triangle RTC, $h = \sqrt{12^2 - \sqrt{80}^2} = \sqrt{64} = 8$.

7. Alan, Bob, Carl and Dick all had stock in Enron, although no two had the same amount, and all lost everything when Enron went belly up. Dick lost less than Bob, and together Dick and Bob lost the same amount as the total lost by Alan and Carl. Together Carl and Dick lost less than Alan. Alan did not lose the most. Order the four from who lost the most to who lost the least.

(a) Alan > Bob > Dick > Carl
(b) Bob > Carl > Alan > Dick
(c) Bob > Carl > Dick > Alan
(d) Carl > Bob > Alan > Dick
(e) Bob > Alan > Carl > Dick

ANSWER: (e)

Method I. Using A, B, C, D for the amounts the corresponding persons have lost, the information translates into the following equations and inequalities:

$$D < B \qquad D + B = A + C \qquad C + D < A,$$

where each of A, B, C and D is positive and A is not the largest of the four numbers. From the last inequality, both C and D are less than A, and since A is not the largest, B has to be the largest number, and A the second largest. Now, if $D > C$, then $D + B > A + C$ contradicting the middle equation above. Therefore, $D < C$, and the order of the losses is $B > A > C > D$.

Method II. We can eliminate suggested answers by comparing to the given information. Answer (a) is clearly incorrect, since Alan did not lose the most, and answer (b) would contradict with Carl and Dick losing less than Alan. Answers (c) and (d) would contradict the fact that Dick and Bob lost the same as Alan and Carl. This gives only (e) as the possible answer; checking shows that this satisfies all requirements.

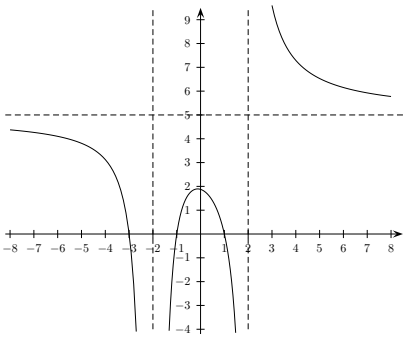
8. Using only the letters from the word WILDCATS with no repetitions allowed in a codeword, how many 4 letter codewords are possible that both start and end with a consonant?

(a) 2,204 (b) 1,680 (c) 900 (d) 360 (e) none of these

ANSWER: (c)

We have 6 choices for the first letter of the codeword (any consonant). since repetitions are not allowed, 5 choices are left for the last letter. That leaves 6 choices for the second letter, and 5 for the third letter of the codeword. We get $6 \times 6 \times 5 \times 5 = 900$ possible codewords.

9. Which of the following functions does the graph of the rational function on the right correspond to?



(a) $\dfrac{5(x^2-1)(x+3)}{(x^2-4)(x+2)}$ (b) $\dfrac{(x^2-1)(x+3)}{(x^2-4)(x+2)}$

(c) $\dfrac{5(x^2-1)(x+3)}{(x^2-4)}$ (d) $\dfrac{(x^2-1)}{(x^2-4)}$ (e) $\dfrac{5(x^2-4)(x+2)}{(x^2-1)(x+3)}$

ANSWER: (a)

The function seems to have a zero (i.e. crosses the x-axis) at -3, -1 and 1. Of all the choices given only (a), (b) and (c) satisfy this. The function in (c) would change sign around the vertical asymptote $x = -2$ and the graph does not, so (c) is not the correct answer either. The horizontal asymptote is the limit of the function at positive or negative infinity, and from the graph, this limit should be about 5. Answer (b) does not satisfy this, so (a) must be the correct answer, and indeed, it satisfies all other requirements.

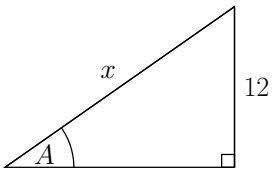
10. Drain pipe number one is observed to drain a pool that is 2/5 full in 3 hours. Number two is observed to drain that same pool when it is 1/4 full in 1.25 hours. How long would it take the two of them, working together, to drain a pool that is 2/3 full?

(a) 1.25 hrs (b) 1.75 hrs (c) 2.25 hrs (d) 2.75 hrs (e) none of these

ANSWER: (e)

We can obtain the hourly rate of each pipe by dividing the amount they drain by the number of hours they have worked. The first pipe can drain 2/15 of the pool in one hour, the second one can drain 1/5 of the pool in one hour. Together they drain $2/15 + 1/5 = 1/3$ of the pool each hour. If the pool was 2/3 full, they need 2 hours to drain it completely. Since this answer is not listed, answer (e) "none of these" is correct.

11. Solve for x in the triangle at the right given that $\tan(A) = 1/2$.

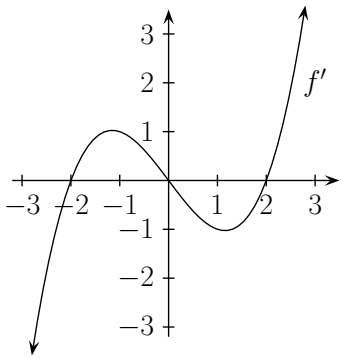


(a) $6\sqrt{5}$ (b) $12\sqrt{5}$ (c) 18 (d) 24 (e) none of these

ANSWER: (b)

If we denote the length of the missing leg by y, then $\tan(A) = 12/y = 1/2$; hence, $y = 24$. Now using the Pythagorean theorem yields $x = 12\sqrt{5}$.

12. The figure on the right shows the derivative of function f. Which of the following are true?
(i) f is decreasing in $(-1, 1)$.
(ii) f has a local minimum at 2.
(iii) f is an odd function.



(a) none are true (b) only (iii) is true (c) only (i) and (iii) are true
(d) only (ii) is true (e) all of them are true

ANSWER: (d)

Let's look at the statements one by one. The first is false, because the function is increasing where it's derivative is positive, and our derivative function has negative values between 0 and 1.

(ii) is true, since the derivative changes sign from negative (decreasing function) to positive (increasing function). The function must have a local minimum at 2.

(iii) is false again. One way to see it is to note that f has a local maximum at 0, which could not happen with an odd function. Actually, if the function were odd, it's derivative would be even (symmetric about the y-axis). Our function cannot be odd.

Thus only (ii) is true, the correct answer is (d).

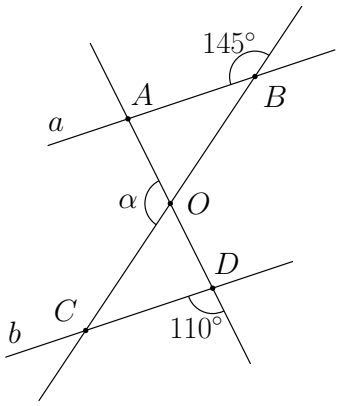
13. The sum of the solutions to the equation $\dfrac{4}{2x+1} - \dfrac{x}{x+2} = 1$ is:

(a) $-\dfrac{1}{2}$ (b) $\dfrac{1}{2}$ (c) $-\dfrac{5}{2}$ (d) $\dfrac{5}{2}$ (e) none of these

ANSWER: (a)

This is a standard textbook problem. Multiplying both sides of the equation by both denominators and rearranging the terms yields the quadratic equation $2x^2 + x - 3 = 0$. Solving it by either the quadratic formula or factoring yields the two roots, 1 and $-3/2$. Substitution into the original equations shows that these are actual solutions. Their sum is $-1/2$.

14. In the figure on the right, lines a and b are parallel. What is the measure of angle α?



(a) 95° (b) 105° (c) 110° (d) 145° (e) none of these

ANSWER: (b)
$\angle ABO = 35°$ by being supplementary to the angle of 145°. The alternate interior angle $\angle OCD = 35°$ as well. Using supplementary angles again, $\angle ODC = 70°$ and from triangle ODC, $\angle COD = 75°$. Thus $\alpha = 105°$.

15. Assuming that all men work at the same rate, if x men can mine y tons of coal in z days, how many tons can b men mine in c days?

(a) $\dfrac{bc}{xz}$ (b) $\dfrac{bc}{xyz}$ (c) $\dfrac{bcx}{yz}$ (d) $\dfrac{bcy}{xz}$ (e) none of these

ANSWER: (d)

One man can mine $\dfrac{y}{x}$ tons of coal in z days, and therefore $\dfrac{y}{xz}$ tons in one day; in turn, b men in c days can mine $\dfrac{bcy}{xz}$ tons.