2005 STATE MATH CONTEST

SOLUTIONS – GRADES 7 – 9

1. Jimmy decided to tell the truth on Mondays, Wednesdays and Fridays, but lie every other day. One day he says: "I will tell the truth tomorrow." What day of the week did he make this statement?

   (a) Monday (b) Thursday (c) Saturday (d) Sunday (e) none of these

   Solution:

   All of Jimmy's truth days are followed by lie days and if he told the truth on Monday, Wednesday or Friday, he would have had to say "I will lie tomorrow." All but one of the lie days are followed by truth days and if he spoke on one of these days he would have had to have said "I will lie tomorrow." This leaves Saturday, a lie day followed by a lie day, where he would have to lie about lying and say "I will tell the truth tomorrow." The correct answer is (c).

2. The planes containing the faces of a cube divide the space into several regions. How many regions?

   (a) 6 (b) 12 (c) 24 (d) 27 (e) 36

   Solution:

   I. Each pair of parallel planes divide the space into three parts. The faces of the cube determine 3 pairs of parallel planes which are not mutually parallel. Therefore, a cube divides the space into $3 \times 3 \times 3 = 27$ regions.

   II. We can count the number of regions by organizing them how they are related to the cube. There is 1 region inside the cube, 6 that are bordered by one face of the cube, 12 that have only a common edge with the cube (one region along each edge of the cube), and 8 that only have a common vertex with the cube. That gives a total of 1+6+12+8=27 regions. The correct answer is (d).

3. In a recent election 65% of the eligible voters voted. The 18–21 year olds made up 26% of the eligible voters and 25% of them voted. What percentage of those who actually voted were 18–21 year olds?

   (a) 8% (b) 10% (c) 12% (d) 14% (e) none of these

   Solution:

   Call n the number of eligible voters. Thus $.65n$ is the total number of people who voted. The number of eligible 18-21 year olds would be $.26n$ and thus the number of this age group who voted would be $(.25)(.26)n = .065n$. The percentage of this age group who actually voted would thus be $\frac{.065n}{.65n} = 10\%$. The correct answer is (b).

4. Which one of the following is a factor of $30x^2 + 11xy - 30y^2$?

(a) $6x + 5y$ (b) $5x - 6y$ (c) $5x + 3y$ (d) $3x - 5y$ (e) none of these

Solution:

One could try to do several long divisions to see whether one of the options divide the given expression, but that is time consuming. It is easier to factor correctly:

$30x^2 + 11xy - 30y^2 = (6x - 5y)(5x + 6y)$, and since neither of these factors or their constant multiples are listed, the correct answer is (e).

5. The function $f(x) = -x^2 + p\,x + q$ has a maximum of 1 and a root (zero) of 2. What is the sum of the values of p that fit those conditions?

(a) -4 (b) 1 (c) 5 (d) 8 (e) 10

Solution:

We are given that $f(2) = 0$, or $-4 + 2p + q = 0$. We also know that the quadratic function $ax^2 + bx + c$ has a maximum at the vertex if a is negative. This vertex is at $x = -b/2a$ which is $-p/2(-1) = p/2$ in our case. $f(p/2) =$ the maximum $= 1$ so we get a second equation of $-(p/2)^2 + p(p/2) + q = 1$. Now we can eliminate q through substitution or by subtracting the two equations to get $p^2 - 8p + 12 = 0$ or $(p - 6)(p - 2) = 0$ from which $p = 6$ or $p = 2$ satisfy the conditions of the problem. The sum of these values is 8, the correct answer is (d).

6. Sam is moving to a colder climate. Right now his car's 20 quart radiator contains a mixture of 20% antifreeze and 80% water. At his new location he will need the mixture to be 50% antifreeze. How many quarts of his current mixture does he need to change for pure antifreeze to reach the required 50% concentration?

(a) 7.5 (b) 2.5 (c) 5 (d) 10 (e) none of these

Solution:

I. Working backwards, he will have 10 quarts of antifreeze and 10 quarts of water in the radiator at the end. The water came from the original 80% mixture, and had to come from 12.5 quart of the original mixture. The remaining 7.5 quart had to be replaced by antifreeze. The correct answer is (a).

II. We can also solve the problem by writing up an equation with x denoting the replaced amount of liquid. Since there are 4 quarts of antifreeze in the car at the beginning, and $0.2x$ gets removed, but x quarts of pure antifreeze is put back resulting in 10 quarts; $4 - 0.2x + x = 10$, from which $x = 7.5$, like before.

7. If we add two angles along every side of a pentagon (going around in order), we get 198°, 210°, 202°, 220°, and 250°. What is the smallest angle of the pentagon?

   (a) 74°   (b) 80°   (c) 88°   (d) 95°   (e) 110°

Solution:

I. Denoting the angles by a, b, c, d, and e (in degrees) we need to find the smallest of them satisfying the equations $a + b = 198$, $b + c = 210$, $c + d = 202$, $d + e = 220$, and $e + f = 250$. Solving the equation system with any standard technique yields $a = 110$, $b = 88$, $c = 122$, $d = 80$ and $e = 140$. The smallest angle of the pentagon is 80°, the correct answer is (b).

II. The second solution is more elegant, using the fact that the sum of the five angles of a pentagon is 540°. Using the previous notation, $a = 540° - (b + c) - (d + e) = 540° - 210° - 220° = 110°$. The other angles can be found similarly.

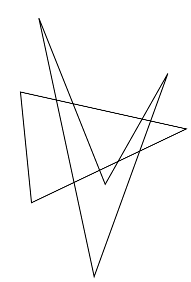
Note: The first solution works for any other polygon, the second solution works only if the polygon has an odd number of sides. Furthermore, the problem gives more than necessary information, the sum of the five pairs must add up to 1080°, twice the angle sum of the pentagon, since every angle is used twice in the total. Knowing all but one of the adjacent angle sums would have been sufficient to find all the angles of any polygon.

8. A triangle and a quadrilateral intersect in finitely many points. What is the most number of intersection points they can have?

   (a) 3   (b) 4   (c) 6   (d) 8   (e) none of these

Solution:

Any side of the quadrilateral can intersect only two sides of the triangle, since every triangle in convex. Since the quadrilateral has four sides, they cannot have more than 8 points of intersections. On the other hand, 8 intersection points are possible as the figure shows. The correct answer is (d).



9. Miss Black, Mr. Crimson, Mrs. Gold, Mr. Green, and Mr. White each own a car that has a color that is the name of one of the other four. Mr. Green's sister is married to the owner of the crimson car. The husband of the owner of the white car carpools with the owner of the green car, who in turn is engaged to Miss Black. Who owns the black car?

   (a) Miss Black (b) Mr. Crimson (c) Mrs. Gold (d) Mr. Green (e) Mr. White

   Solution:

   From the third sentence, the owner of the white car has a husband, and the only married woman in the company is Mrs. Gold, she owns the white car. Mr. Green's sister is married to the owner of the crimson car, and that can't be Mr. Crimson or Mr. Green, the only remaining man is Mr. White, he owns the crimson car. Miss Black's car is not black (by the first rule, nobody owns a car of the same color as their name), not crimson or white (they already have owners), and not green (its owner is a man). Miss black owns the gold car. What color is Mr. Green's car? The gold, white and crimson cars have other owners, he must own the black car. The correct answer is (d).

10. What is the smallest value of n such that the factorial of n is divisible by 414?

   (a) 6 (b) 33 (c) 207 (d) 414 (e) none of these

   Solution:

   The key is to find the factorization of $414 = 2 \cdot 3^2 \cdot 23$. Since 23 is prime, we need at least 23! to have it divisible by 414, and since 23! has plenty of 2's and 3's as well, that is the smallest factorial divisible by 414. The correct answer is (e).

11. Sets A, B, C and D are all subsets of quadrilaterals. A is the set of rhombi, B is the set of rectangles, C is the set of parallelograms, and D is the set of kites. What is the set $(A \cap B) \cup (C \cap D)$?

   (a) squares (b) rectangles (c) parallelograms (d) rhombi (e) none of these

   Solution:

   A rhombus has four equal sides, a rectangle has four $90°$ angles, thus $A \cap B = \{\text{squares}\}$. A parallelogram that has two pairs of equal adjacent sides must have all sides equal, and therefore $C \cap D = \{\text{rhombi}\}$. Thus,

   $(A \cap B) \cup (C \cap D) = \{\text{squares}\} \cup \{\text{rhombi}\} = \{\text{rhombi}\}$. The correct answer is (d).

12. The numbers 1, 2, 3,..., 20 are written on the blackboard. Then one will erase two of the numbers a and b and write $a+b-1$ on the board instead. After repeating this step 19 times, there will be only one number on the board. What is that number?

(a) 190 (b) 191 (c) 209 (d) 210 (e) none of these

Solution:

The problem seems to imply that the remaining number does not depend on the order of erasing the numbers. If one is willing to believe this, trying any convenient order and recognizing a pattern can lead to the correct solution. We'll provide a solution that actually proves that the final number does not depend on what order we erase the numbers.

Notice that in each step the sum of the numbers on the board decreases by 1. The sum of the original numbers can be found easily by pairing the first and last numbers to make 21, then second to second to last to make 21, and so on. As we made ten pairs, the sum of the numbers was $10 \times 21 = 210$. This sum in 19 steps will then decrease to 191. That is the last number on the board. The correct answer is (b).

Note: We could have used a formula for arithmetic sequences to add the original 20 numbers, or simply just do it in writing or in our head, it's only 20 terms.

13. Suppose you could trade your current job for a new job that pays 40% more per hour, but would require 30% fewer hours of work per week. How would your new weekly salary compare to your current one?

(a) 10% more (b) 10% less (c) 2% more (d) 2% less (e) none of these

Solution:

If we only increased the hourly rate, the salary would increase by a factor of 1.4. Similarly, reducing the hours results in a decrease by a factor of 0.7. Their combined effect will change the salary by a factor of $1.4 \times 0.7 = 0.98$, that is a 2% decrease. The correct answer is (d).

14. What is the sum of the solutions of the equation $|2 - |1-x|| = 1$?

(a) -2 (b) 0 (c) 4 (d) 6 (e) none of these

Solution:

To satisfy the equation, $2-|1-x| = 1$ or -1. Correspondingly, $|1-x| = 1$ or $|1-x| = 3$. The first equation is satisfied by $x = 0$ and $x = 2$, while the second by $x = -2$ and $x = 4$. The sum of these four solutions is 4, the correct answer is (c).

15. The graphs of $y = 2^{4x^2+2x+4}$ and $y = 4^{8x^2+2x+1}$ intersect in two points: (x_1, y_1) and (x_2, y_2). The value of x_1x_2 is:

(a) -1/2 (b) 1/3 (c) -1/6 (d) 1/8 (e) none of these

Solution:

The graphs will intersect where $2^{4x^2+2x+4} = 4^{8x^2+2x+1}$ or equivalently $2^{4x^2+2x+4} = 2^{16x^2+4x+2}$. Since the exponential function is one-to-one, this can only happen if $4x^2 + 2x + 4 = 16x^2 + 4x + 2$ or $6x^2 + x - 1 = 0$. Either factoring or the quadratic formula now yields $x_1 = -1/2$ and $x_2 = 1/3$, and the product of these two numbers is $-1/6$, the correct answer is (c).

16. What is the value of $\dfrac{a+b}{a-b} - \dfrac{a-b}{a+b}$, if $a = x + y$, $b = x - y$ and $x, y \neq 0$?

(a) $\dfrac{x^2-y^2}{xy}$ (b) $\dfrac{x^2-y^2}{2xy}$ (c) 1 (d) $\dfrac{x^2+y^2}{xy}$ (e) $\dfrac{x-y}{xy}$

Solution:

Since $a + b = 2x$ and $a - b = 2y$ we can simplify

$\dfrac{a+b}{a-b} - \dfrac{a-b}{a+b} = \dfrac{2x}{2y} - \dfrac{2y}{2x} = \dfrac{x^2-y^2}{xy}$, the correct answer is (a).

17. If a right circular cylinder of height H and radius R were 3/4 full of water, how many times could a right circular cylinder of height $H/2$ and radius $R/2$ be filled with the water from the large cylinder?

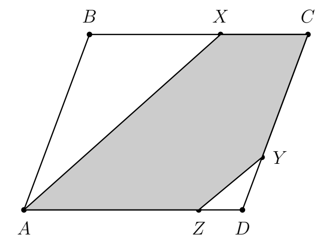
(a) 2 (b) 4 (c) 6 (d) 8 (e) none of these

Solution:

I. Using the formula for the volume of the cylinder, we have $\frac{3}{4}R^2\pi H$ water to fill cylinders of volume $\left(\frac{R}{2}\right)^2\pi\frac{H}{2}$. Dividing the first expression with the second we get 6, that is how many times the small cylinder can be filled with the water. The correct answer is (c).

II. We don't need to know the volume formulae to solve the problem. Notice that the small cylinder is similar to the large one with a ratio of similarity of 1/2. Since volume is proportional to the cube of this scaling factor, the small cylinder is 1/8 the volume of the big one. We can fill the small one $\frac{3}{4} \div \frac{1}{8} = 6$ times.

18. Quadrilateral $ABCD$ is a parallelogram with area 120. Point X cuts BC so that $BX : XC = 3 : 2$. Point Y cuts CD so that $CY : YD = 2 : 1$, and Z cuts AD so that $AZ : ZD = 3 : 1$. What is the area of the pentagon $AXCYZ$?



(a) 41 (b) 47 (c) 63 (d) 73 (e) 79

Solution:

Denote by b the length of the base of the parallelogram (AD), and h be the corresponding height. The area of a parallelogram is $bh = 120$. If we look at the areas of the non-shaded triangles: area($\triangle ABX$) $= \frac{1}{2}\frac{3}{5}bh = \frac{3}{10}bh = 36$ and area($\triangle ZDY$) $= \frac{1}{2}\frac{1}{4}b\frac{1}{3}h = \frac{1}{24}bh = 5$. This gives us an unshaded area of $36 + 5 = 41$. So the shaded area is $120 - 41 = 79$. The correct answer is (e).

19. Read the following 5 statements carefully:

(i) Statement (ii) is true.

(ii) At most one of these 5 statements is true.

(iii) All 5 of these statements are true.

(iv)

(v)

The last two statements are printed in invisible ink. Which of the statements are true?

(a) only (i) (b) only (iv) and (v) (c) all of them (d) none of them (e) cannot be determined

Solution:

If the first statement (i) were true, it would make (ii) true, but then we would have two true statements already, and that contradicts (ii). Therefore, (i) must be false, and so is (ii). Clearly (iii) is false, and the only way to have at least two correct statements (remember, (ii) is false) is to have (iv) and (v) both true. The correct answer is (b).

20. How many three digit prime numbers are there with the property that the product of the three digits is 10?

(a) 1 (b) 2 (c) 3 (d) 6 (e) none of these

Solution:

What are the three digits? The only three digits that multiply to 10 are 1,2,5. Neither 2 nor 5 can be in the last position (the number would be divisible by 2 or 5), so the last digit is 1. The resulting two numbers 251 and 521 need to be checked separately, and both turn out to be primes. The correct answer is (b).

21. Roll 3 dice, and multiply the three numbers showing. What is the probability that the product is even?

(a) $\frac{1}{2}$ (b) $\frac{2}{3}$ (c) $\frac{3}{4}$ (d) $\frac{7}{8}$ (e) none of these

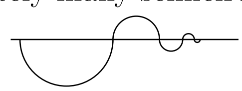
Solution:

The key is to find the probability of the complement, the event that the product is odd. That can only happen when all three dice show an odd number, and it has a probability of $(1/2)^3 = 1/8$. Then

$P(\text{product is even}) = 1 - P(\text{product is odd}) = 1 - \dfrac{1}{8} = \dfrac{7}{8}.$

The correct answer is (d).

22. What is the length of the curve that is built by infinitely many semicircles with radius 1, 1/2, 1/4, etc. according to the picture:



(a) 2π (b) $\dfrac{7\pi}{4}$ (c) $\dfrac{\pi}{6}$

(d) $\dfrac{8\pi}{3}$ (e) none of these

Solution:

The first semicircle has length of π, the second has length of $\frac{1}{2}\pi$, the third has length of $\frac{1}{4}\pi$, etc. The total length is

$\pi + \dfrac{1}{2}\pi + \dfrac{1}{4}\pi + \ldots = \pi(1 + \dfrac{1}{2} + \dfrac{1}{4} + \ldots) = 2\pi,$

where the last summation can be done using the formula for the sum of a geometric sequence, or simply by visualizing the pieces next to each other adding up to 2. The correct answer is (a).

23. In a triathlon competition one contestant ran for a distance at 12 miles per hour, then swam for one-third that distance ar 4 miles per hour and then, finally, bicycled for 4 times the first distance at 20 miles per hour. Which one of the following is closest in value to his average speed over the whole course?

(a) 7 mph (b) 9 mph (c) 11 mph (d) 13 mph (e) 15 mph

Solution:

Denote the distance of the of running by x. Then the total distance they competed is $x + x/3 + 4x$, while the total time the contestant took is $\frac{x}{12} + \frac{x/3}{4} + \frac{4x}{20} = 11x/30$. His average speed is $\frac{x+x/3+4x}{11x/30} = 160/11 \approx 14.5$ mph. The correct answer is (e).

24. Art, Bob, Chuck, and Dean have started collecting baseball cards. Together they have 84 cards. Dean has three quarters as many as Bob has. Art has the average of the other three, and Bob has twice as many as Chuck. How many cards does Dean have?

(a) 28 (b) 14 (c) 16 (d) 18 (e) 21

Solution:

I. Let Bob have b baseball cards. Then Dean has $\frac{3}{4}b$, Chuck has $\frac{1}{2}b$ and Art has $\frac{1}{3}(b + \frac{3}{4}b + \frac{1}{2}b) = \frac{3}{4}b$. Solving now $b + \frac{3}{4}b + \frac{1}{2}b + \frac{3}{4}b = 84$ yields $b = 28$, so Dean has $\frac{3}{4} \cdot 28 = 21$ cards. The correct answer is (e).

II. The second solution uses no algebra at all. If Art has the average of the other three, he must have a quarter of the total cards, 21 cards. The other 63 cards are divided among Bob, Chuck, and Dean. Clearly, Bob has the most of them, and Chuck has the least. If Bob gave a quarter of his cards to Chuck, all of them had an equal amount, therefore Dean had 21 cards all the time.

25. Sam runs 1000 meters in an average of 160 seconds. Harry averages 180 seconds. Which one of the following is closest to the number of meters of advantage Sam should give Harry in order to finish a 1000 meter race at the same time.

(a) 100 (b) 110 (c) 120 (d) 130 (e) 140

Solution:

The easiest way to find how much advantage Harry needs is to let him start running, and make Sam start 20 seconds later. Wherever Harry is at this moment, they'll finish the race at the same time, another 160 seconds later. In 20 seconds Harry raced 1/9 of the race, about 111 meters, which is closest to 110 meters of the listed numbers. The correct answer is (b).

26. Sequence $x_1, x_2, x_3, \ldots$ satisfies $x_1 + x_2 + \cdots + x_n = n^3$ for all natural numbers n. A formula for x_n is:

(a) $3n^3 - 3n^2 + 1$ (b) $3n^2 + n + 1$ (c) $3n^2 - 3n + 1$
(d) $n^3 - 3n^2 + 3n - 1$ (e) none of these

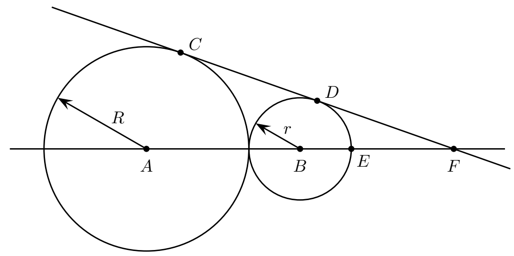
Solution:

One needs to know (or calculate by multiplying out) that $(n-1)^3 = n^3 - 3n^2 + 3n - 1$, and then notice that

$$x_n = (x_1+x_2+\ldots+x_n)-(x_1+x_2+\ldots+x_{n-1}) = n^3-(n-1)^3 = n^3-(n^3-3n^2+3n-1) = 3n^2 - 3n + 1$$

The correct answer is (c).

27. Two circles are tangent to each other, have radii of R and r, with $r < R$, and have centers at A and B, respectively. If CD is a common tangent line to the circles (see picture), what is the length of EF?



(a) $\dfrac{2r^2}{R-r}$ (b) $\dfrac{2R^2}{R-r}$ (c) $\dfrac{r^2}{R-r}$

(d) $\dfrac{R^2}{R-r}$ (e) none of these

Solution: Triangles ACF and BDF are similar, because they have a common angle at F and right angles at both C and D. Therefore their sides are proportional, $AC/BD = AF/BF$ replacing these distances with the known radii, $\frac{R}{r} = \frac{R+2r+EF}{r+EF}$. Solving this equation for yields $EF = \frac{2r^2}{R-r}$. The correct answer is (a).

28. If the perimeter of a rectangle is 40 feet, and the area of it 75 ft^2, how many feet long is the diagonal of the rectangle?

(a) 15 (b) $\sqrt{15}$ (c) $5\sqrt{10}$ (d) $25\sqrt{10}$ (e) none of these

Solution: Although the length (l) and width (w) can easily be found by trial and error methods, here we'll discuss an algebraic approach. Using the area and perimeter given, $2l + 2w = 40$ and $lw = 75$. From the first equation $w = 20 - l$, substituting this into the second we get $l(20 - l) = 75$. This simplifies to $l^2 - 20l + 75 = 0$ or $(l-5)(l-15) = 0$. The two solutions give us possible lengths of 5 and 15 feet, but the corresponding widths are 15 and 5 feet, so we really have only one solution.

To find the diagonal one needs to use the Pythagorean theorem for a right triangle with legs of 5 and 15, we get a diagonal of $\sqrt{5^2 + 15^2} = \sqrt{250} = 5\sqrt{10}$. The correct answer is (c).

29. Which is true? The six-digit number *abcabc* is always divisible by

(a) 101 (b) 91 (c) 99 (d) 81 (e) 33

Solution: Since $(abcabc) = 1001 \cdot (abc) = 91 \cdot 11 \cdot (abc)$ we can see that the number is always divisible by 91. The correct answer is (b).

Note: Just to make sure, one can check that the other numbers don't always divide a six-digit number of the required form. Try 100100 for example.

30. A rectangular picture has an area of 120 in^2. If the ratio of the length to the width is 6:5, what is the perimeter of the picture?

(a) 22 in (b) 42 in (c) 44 in (d) 52 in (e) none of these

Solution: One needs to find the length and width of the picture first. While an easy equation provides $l = 12$ and $w = 10$, these numbers can also be found by trial and error, or taking a 6 by 5 rectangle and scaling it up by the necessary factor of 2. Either way the perimeter now is $2(10 + 12) = 44$ inches. The correct answer is (c).