2005 STATE MATH CONTEST

GRADE 10-12 PRETEST SOLUTIONS

1. Eight swimmers competed in lanes 1-8 in a race. Those swimming in lanes 5, 6 and 7 finished in the last three places in some order. Three swimmers who are swimming in adjacent lanes finished at the same place as their lane number. The other five swimmers finished in different places as their lane numbers, and only one of them had both numbers (lane and place) even, with the other four one of the numbers was even, the other was odd. Which lane did the winner swim on?

   (a)* lane 1 (b) lane 2 (c) lane 3 (d) lane 4 (e) lane 8

   Solution:

   The key is to figure out which three swimmers on adjacent lanes finished on the same place as their lane numbers. It couldn't be any of swimmers on lanes 5, 6, 7, or 8, because the swimmers in lanes 5 and 8 have moved in the order for sure (5 into one of the last 3 places, 8 away from the last place). If swimmers 2, 3, and 4 finished in places 2, 3, and 4 in that order, then swimmers 1, 5, 6, 7, 8 are the other five who did not finish on the place of their lane numbers. But out of them 1, 5, 7, and one more (6 or 8) would have to place on even numbers, while we only have two even positions (6th and 8th) available.

   That leaves only one possibility, namely that the swimmers on lanes 1, 2, and 3 place as first second and third in that order. The correct answer to the question is (a).

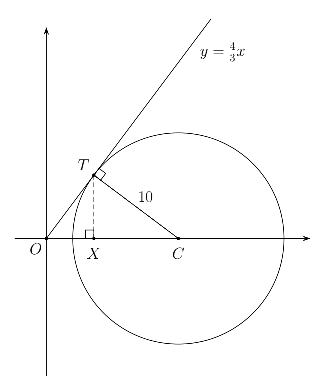
   Note: Although not part of the problem, we can almost completely determine the order of the rest of the swimmers too, it must have been 1, 2, 3, 8, 4, 5/7 ,6, 7/5 where those on lanes 5 and 7 can take one of places 6 and 8.

2. A circle of radius 10 is centered on the the x-axis, on the positive side of it, and is tangent to $y = \frac{4}{3}x$. The x-coordinate of the center of that circle is closest in value to which one of the following?

   (a)* 13 (b) 14 (c) 15 (d) 16 (e) 17

   Solution:

   I. The first solution uses simple geometry. From the equation of the tangent line OT we get $TX = (4/3)OX$. Using Pythagorean theorem $OT^2 = OX^2 + (\frac{4}{3}OX)^2$, from which $OT/TX = 5/4$. Triangles OXT and OTC are similar, since both are right triangles with a common acute angle. From the ratio of the corresponding sides $OC/TC = OT/TX = 5/4$, and since $TC = 10$, $OC = 12.5$. Of the listed numbers, 12.5 is closest to 13, the correct answer is (a).



II. Here is second solution using algebra. The equation of the circle $y^2 + (x - c)^2 = 10^2$, combined with the equation of the line $y = 4/3x$, is a system of equations that must have a unique solution. Substituting the second into the first, and simplifying we get $25x^2 - 18cx + 9c^2 - 900 = 0$. We need to find c so that this equation has only one solution. For that, the discriminant has to be 0, so $324c^2 - 100(9c^2 - 900) = 0$ from which we get one positive solution, $c = 12.5$. Again, the correct answer is (a).

3. Which of the following functions satisfy $f(a + b) = f(a)f(b)$?

   (i) $f(x) = 1$ (ii) $f(x) = 3^x$ (iii) $f(x) = |x|$ (iv) $f(x) = \sqrt{x}$

   (a) only (ii) (b) only (ii) and (iv) (c)* only (i) and (ii) (d) only (i) and (iv) (e) none of these

   Solution:

   The constant 1 function obviously satisfies the condition, since $1 = 1 \times 1$. The exponential function does too, because $3^{a+b} = 3^a 3^b$. The absolute value does not satisfy the condition in general, for instance $|3 + (-2)| \neq |3| \times |-2|$. Neither does the square root function, for example $\sqrt{9 + 16} \neq \sqrt{9}\sqrt{16}$.

   Only (i) and (ii) satisfy the required property, the correct answer is (c).

4. When $\left(x^{1/4} - x^{2/3}\right)^7$ is multiplied out and simplified one of the terms has the form Kx^3 where K is a constant. Find K.

   (a) 7 (b) -7 (c) 35 (d)* -35 (e) none of these

   Solution:

   I. If we are using the binomial theorem to write out the product, the critical term will have the form of $C(7, r)(x^{1/4})^{7-r}(-x^{2/3})^r = Kx^3$. From this $\dfrac{7-r}{4} + \dfrac{2r}{3} = 3$, which solves to $r = 3$. Then $K = -C(7, 3) = -\dfrac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = -35$. The correct answer is (d).

   II. It is possible (but not recommended) that somebody could multiply out the terms to obtain the correct answer without using the binomial theorem. One would get the expression $-x^{56/12} + 7x^{51/12} - 21x^{46/12} + 35x^{41/12} - 35x^{36/12} + 21x^{31/12} - 7x^{26/12} + x^{21/12}$. The solution is clearly $K = -35$.

5. Urn A contains 9 red balls and 11 white balls. Urn B contains 12 red balls and 3 white balls. One is to roll a single fair die. If the result is a one or a two, then one is to randomly select a ball from urn A. Otherwise one is to randomly select a ball form urn B. What is the probability of obtaining a red ball?

   (a)* $\dfrac{41}{60}$ (b) $\dfrac{19}{60}$ (c) $\dfrac{21}{35}$ (d) $\dfrac{35}{60}$ (e) none of these

   Solution:

Using the Law of Total Probabilities (or common sense),

$$P(\text{red ball}) = \frac{2}{6} \cdot \frac{9}{20} + \frac{4}{6} \cdot \frac{12}{15} = \frac{41}{60}$$

The correct answer is (a).

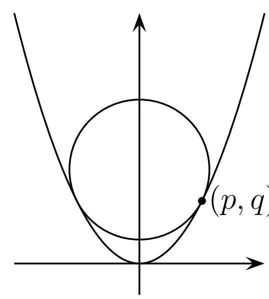
6. What is the smallest perfect square number that is divisible by both 45 and 72?

   (a) 360 (b) 1,600 (c) 3,240 (d)* 3,600 (e) none of these

   Solution:

   First let's examine the prime factorizations of the two numbers: $45 = 3 \cdot 3 \cdot 5$ and $72 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot$. Any common multiple must contain at least three factors of 2, two factors of 3 and one factor of 5. In addition to make the common multiple a perfect square, every prime factor must have an even multiplicity, the smallest such number is $2^4 \cdot 3^2 \cdot 5^2 = 3,600$. The correct answer is (d).

7. A circle of radius $4\sqrt{5}$ has its center on the y-axis and is tangent to the parabola $y = \frac{1}{8}x^2$. Point (p, q) is their common point in the first quadrant (see picture). What is the value of $p + q$?



   (a) 14 (b) 15 (c)* 16 (d) 17 (e) 18

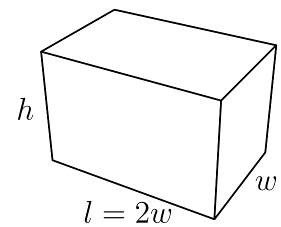
   Solution:

   Since our point lies on the parabola, $q = p^2/8$. The derivative of the function at a point of the parabola $(x, \frac{1}{8}x^2)$ is $x/4$ and at (p, q) this value, $p/4$, is the slope of the common tangent line to both the circle and the parabola. The perpendicular line to this tangent line at (p, q) must have a slope of $-4/p$ and goes through the center of the circle $O = (0, z)$. From these two points and the slope we get $\dfrac{z - p^2/8}{0 - p} = -\dfrac{4}{p}$ from which $z = \dfrac{p^2}{8} + 4$. On the other hand the distance between O and the point of tangency (p, q) must be the radius of the circle, so $(p^2/8 + 4 - p^2/8)^2 + (0 - p)^2 = (4\sqrt{5})^2$. Solving this equation we get one positive solution: $p = 8$. The corresponding y value is $q = p^2/8 = 8$, and the sum $p + q = 16$. The correct answer is (c).

8. A rectangular box having no top is to be twice as long as it is wide and is to hold 12 ft^3.Which of the following is closest to the height (in feet) of the box of that type having the minimum surface area?

   (a) 1/3 (b) 1/2 (c) 2/3 (d) 3/4 (e)* 1

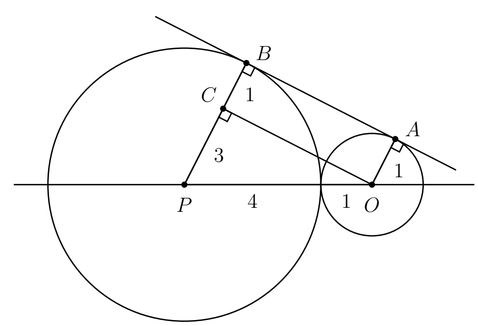
Solution:
Let the dimensions of the box be w, $2w$, and h in feet. From the volume, we get $2w^2h = 12$, or $h = 6/w^2$. The surface area of the box is now $2w^2 + 6w\frac{6}{w^2}$. To minimize this function, the function must have a critical point, either a point where it is undefined or where the derivative is 0. Since we are looking for positive solutions, $w = 0$ is not of our interest, and from the derivative of the function we get $4w - 36/w^2 = 0$, from which $w = \sqrt[3]{9}$. Since the second derivative of the surface area function $4 + 72/w^3$ is positive at $w = \sqrt[3]{9}$, this value of w does provide a minimum for the surface area. Thus $h = 6/w^2 = 2/\sqrt[3]{3} > 1$. The correct answer is (e).



9. Two circles of radius 1 and 4 units are tangent to each other externally. The length of the part of the common tangent line between the points of tangency is closest to which of the following?

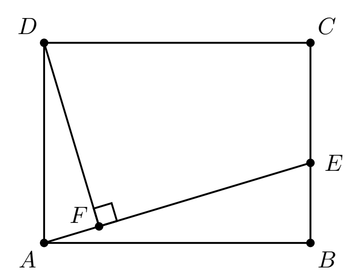
(a) 3.6 (b)* 3.9 (c) 4.2 (d) 4.5 (e) 4.8

Solution:
Let the circles have centers at O and P and the common tangent segment in question be AB as shown in the picture. Both OA and PB are perpendicular to AB, the tangent segment. Now draw a line through O that is parallel with AB, and label the point where it intersects PB with C. Triangle OPC is a right triangle with sides $OP = 5$, and $PC = 3$. From the Pythagorean theorem $OC = 4$, and since $ABCO$ is a rectangle, $AB = OC = 4$; which is closest to 3.9. The correct answer is (b).



10. In rectangle $ABCD$ (see figure) side AB is one third longer than side BC. Point E divides BC such that EC is twice as long as BE. Point F is chosen on AE to make AE is perpendicular to DF. Find the ratio of the area of triangle ADF to the area of rectangle $ABCD$.

(a) 4/41 (b) 2/21 (c)* 3/34 (d) 5/48 (e) none of these



Solution:

For convenience, let $AB = 4$. Then $BC = 3$, and $BE = 1$. Clearly, the area of $\triangle BEA$ is 1/6 of the area of the rectangle. From Pythagorean theorem $AE = \sqrt{17}$. Using the alternate interior angles $\angle DAF = \angle BEA$, the right triangles $\triangle ABE \sim \triangle DFA$. The ratio of similarity, from the hypotenuses is $3/\sqrt{17}$, so the area of $\triangle DAF$ is 9/17 of the area of $\triangle BEA$, and 3/34 of the area of the rectangle. The correct answer is (c).

11. One root of $mx^2 - 10x + 3 = 0$ is two thirds of the other root. What is the sum of the roots?

(a) 3/2 (b) 5/2 (c) 7/2 (d) 1/4 (e)* 5/4

Solution:

I. One approach is to use the quadratic formula to find the two roots: $x_1 = \frac{10+\sqrt{100-12m}}{2m}$ and $x_2 = \frac{10-\sqrt{100-12m}}{2m}$. Next we can substitute $x_1 = 2x_2/3$ or $x_2 = 2x_1/3$, since we don't know which root is 2/3 of the other. From the first substitution we don't get any real solutions, from the second we get $m = 8$. Then $x_1 = 3/4$ and $x_2 = 1/2$, the sum of those roots is 5/4. The correct answer is (e).

II. A second approach is to rewrite $mx^2 = -10x+3 = 0$ as $x^2 - \dfrac{10x}{m} + \dfrac{3}{m} = 0$. Our roots are r and $2r/3$, so the factored form of the quadratic equation is $(x-r)(x-2r/3) = x^2 - \dfrac{5rx}{3} + \dfrac{2r^2}{3}$. From this $\dfrac{-5r}{=}\dfrac{-10}{m}$ and $\dfrac{2r^2}{3} = \dfrac{3}{m}$. Solving this system of equations leads to the same result as before.

12. Given that the fourth term of a geometric sequence is 108 and the 7th term is 2916, what is the first term of the sequence?

(a) 3 (b)* 4 (c) 8 (d) 12 (e) none of these

Solution:

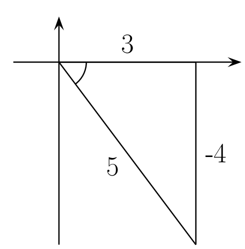
I. The ratio of the 7th and 4th term is the same as the 4th and 1st terms (the cube of the ratio of the sequence). If the first term of the sequence is x, then solving $\dfrac{x}{108} = \dfrac{108}{2916}$ gives $x = 4$ very easily. The correct answer is (b).

II. A little more formally, in the geometric sequence (a_n) $a_7 = x \cdot q^6 = 2916$, where x is the first term of the sequence and q is the ratio. Similarly, $a_4 = x \cdot q^3 = 108$. Dividing the first equation by the second, we get $q^3 = 27$, from which $q = 3$. Now, by substituting back into either of the original equations we easily get $x = 4$.

13. What is $\sin(\tan^{-1}(-4/3))$ equal to?

(a)* -4/5 (b) 4/3 (c) 3/5 (d) -3/4 (e) none of these

Solution:
The figure represents the angle between $-\pi/2$ and $\pi/2$ (the range of $\tan^{-1}$) having a tangent of $-4/3$. The sine of this angle is clearly $-4/5$. The correct answer is (a).



14. Which of the following are true for all $x > 0$, $y > 0$, $b > 0$, $b \neq 1$, and all positive integers n?

(i) $(x+y)(x-y) = x^2 - y^2$ (ii) $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$ (iii) $\log_b(x^n) = n \log_b x$

(iv) $\left(\dfrac{x}{y}\right)^n = \dfrac{x^n}{y^n}$ (v) $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$

(vi) $\log_b(x+y) = \log_b(x) + \log_b(y)$ (vii) $n^{x+y} = n^x n^y$

(a) only (i), (iii), (iv) and (vii) are true (b) only (i), (iv) and (vii) are true

(c) all are true (d)* all except (ii) and (vi) are true (e) all except (ii) are true

Solution:

Except for (ii) and (vi), all the statements are standard identities or ones that can easily be verified by multiplying the terms out. To see that (ii) and (vi) are false we can consider the following counterexamples:

(ii) $5 = \sqrt{9+16} \neq \sqrt{9} + \sqrt{16} = 7$.

(vi) $1 = \log_2(1+1) \neq \log_2 1 + \log_2 1 = 0$.

The correct answer is (d).

15. A force of magnitude 15 lbs makes an angle of 60° with a force of magnitude 8 lbs. Which one of the following is closest to the magnitude of the resultant of those two forces?

(a) 18 lbs (b)* 20 lbs (c) 22 lbs (d) 24 lbs (e) 26 lbs

Solution:

To add the two vectors we need to position them "head-to-tail". Adding the vectors of length 15 and 8 that enclose a 60° angle produces a triangle with a 120° angle (see picture). Using the law of cosines,

$$x^2 = 15^2 + 8^2 - 2 \cdot 15 \cdot 8 \cdot \cos 120^\circ = 225 + 64 - 240 \cdot (-1/2) = 401$$

from which $x = \sqrt{409}$. This is very close to 20, so the correct answer is (b).