2005 STATE MATH CONTEST

SOLUTIONS – GRADES 10 – 12

1. What is the sum of the solutions of the equation $\sqrt{4x+15}-3=\sqrt{2x-1}$?

   (a) 11 (b) 3 (c) 9 (d) 12 (e) 5

   Solution:

   First we square both sides of the equation until we have no more square roots left on either side. Notice that this step may produce extraneous solutions, we'll need to check at the end.

   $(\sqrt{4x+15}-3)^2=(\sqrt{2x-1})^2 \Longrightarrow 4x+15-6\sqrt{4x+15}+9=2x-1 \Longrightarrow$
   $(2x+25)^2=(6\sqrt{4x+15})^2 \Longrightarrow 4x^2+100x+625=36(4x+15) \Longrightarrow 4x^2-44x+85=0.$

   We can proceed two different ways now.

   I. We can use either factoring or the quadratic formula to find the two solutions of the quadratic equation, which are $8\frac{1}{2}$ and $2\frac{1}{2}$. Substituting these into the original equation verifies that they are true solutions of it. There sum is 11, the correct answer is (a).

   II. Recall that the solutions x_1 and x_2 of the quadratic equation $ax^2+bx+c=0$ satisfy $x_1+x_2=-\frac{b}{a}$ and $x_1x_2=\frac{c}{a}$. Since $x_1x_2=\frac{85}{4}$ is positive, x_1 and x_2 have the same sign. Since $x_1+x_2=-\frac{-44}{4}=11$ is positive, both x_1 and x_2 must be positive and so neither is an extraneous solution to our original equation. Hence, the sum of the two solutions is 11. The correct answer is (a).

2. The planes containing the faces of a cube divide the space into several regions. How many regions?

   (a) 6 (b) 12 (c) 24 (d) 27 (e) 36

   Solution:

   I. Each pair of parallel planes divide the space into three parts. The faces of the cube determine 3 pairs of parallel planes which are not mutually parallel. Therefore, a cube divides the space into $3\times3\times3=27$ regions.

   II. We can count the number of regions by organizing them how they are related to the cube. There is 1 region inside the cube, 6 that are bordered by one face of the cube, 12 that have only a common edge with the cube (one region along each edge of the cube), and 8 that only have a common vertex with the cube. That gives a total of 1+6+12+8=27 regions. The correct answer is (d).

3. How many digits (in base 10) do 2^{2005} and 5^{2005} have altogether?

   (a) 2006 (b) 2005 (c) 4010 (d) 2004 (e) none of these

   Solution:

   Let 2^{2005} have x digits, and 5^{2005} have y digits. Since neither number is exactly a power of 10, we know that $10^{x-1} < 2^{2005} < 10^x$ and $10^{y-1} < 5^{2005} < 10^y$. If we multiply these inequalities (all sides are positive), we get $10^{x+y-2} < 2^{2005}5^{2005} < 10^{x+y}$. Notice that the product in the middle is 10^{2005}, therefore $x+y-2 < 2005 < x+y$, and $x+y = 2006$. The correct answer is (a).

4. A convoy is traveling at 30 miles per hour. A messenger on a motorcycle traveling at 50 miles per hour can go from one end of the convoy to the other and back again in 23 minutes. About how many miles long is the convoy?

   (a) 8.2 (b) 5.6 (c) 7.1 (d) 6.5 (e) 6.1

   Solution:

   Compared to the convoy the motorcycle travels at $50 - 30 = 20$ mph in one direction and $50 + 30 = 80$ mph in the other direction. Denoting the length of the convoy by x miles, the time of the total trip is then $\dfrac{x}{80} + \dfrac{x}{20} = \dfrac{23}{60}$. Solving this yields $x = \dfrac{92}{15}$ miles, the correct answer is (e).

5. In a drawer there are 18 socks, 5 black, 6 white and 7 green. If one morning one pulls out two socks at random (with closed eyes), what is the probability of obtaining a matching pair?

   (a) 46/153 (b) 13/36 (c) 4/105 (d) 55/162 (e) none of these

   Solution:

   I. How many ways are there to choose two socks out of the drawer at random? There are $\binom{18}{2} = \frac{18\cdot 17}{2} = 153$ ways. Out of these, the number of ways we can get two black socks is $\binom{5}{2} = 10$. Similarly, two white or two green socks we can get $\binom{6}{2} = 15$ and $\binom{7}{2} = 21$ ways, respectively. The probability of pulling two matching socks is $\frac{10+15+21}{153} = \frac{46}{153}$, the correct answer is (a).

   II. Another way of solving the problem is to find the probability of pulling two black socks first.

   $P(\text{two black}) = P(\text{first is black}) \cdot P(\text{second is black given the first was black}) = \frac{5}{18} \cdot \frac{4}{17}$

   Since obtaining matching pairs of different colors are disjoint events, and their probability can be calculated similarly,

   $P(\text{matching pair}) = P(\text{two blacks}) + P(\text{two whites}) + P(\text{two greens}) =$

   $= \frac{5}{18} \cdot \frac{4}{17} + \frac{6}{18} \cdot \frac{5}{17} + \frac{7}{18} \cdot \frac{6}{17} = \frac{46}{153}.$

6. The negation of the statement "No hungry man is happy" is:

(a) All hungry man are happy
(b) All hungry man are unhappy
(c) At least one hungry man is happy
(d) No hungry man is unhappy
(e) At least one hungry man is unhappy

Solution:

The statement claims that every man who is hungry is also unhappy. The negation of this is that there is at least one hungry man who is happy. The correct answer is (c).

7. If $x-1$, x^2-1, x^3-1, and x^4-1 all are factors of a polynomial $p(x)$, what is the smallest possible degree of $p(x)$?

(a) 5 (b) 6 (c) 7 (d) 10 (e) 12

Solution:

We need to factor the given polynomials into irreducible factors; $x^2-1=(x-1)(x+1)$, $x^3-1=(x-1)(x^2+x+1)$, $x^4-1=(x-1)(x+1)(x^2+1)$. Our $p(x)$ must contain each of these factors, $(x-1)(x+1)(x^2+x+1)(x^2+1)$ is such a polynomial with smallest possible degree of 6. The correct answer is (b).

8. What is the real part of $(1+i)^{50}$?

(a) 0 (b) 2^{25} (c) -2^{25} (d) 2^{50} (e) -2^{50}

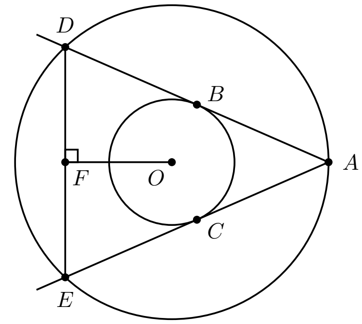
Solutions:

I. $(1+i)^{50}=[(1+i)^2]^{25}=(1+2i+i^2)^{25}=(2i)^{25}=2^{25}i^{25}=2^{25}(i^4)^6 i=0+(2^{25})i$. The correct answer is (a).

II. Recall that $a+bi=re^{i\theta}=r(\cos\theta+i\sin\theta)$ where $r=\sqrt{a^2+b^2}$, $\cos\theta=\frac{a}{r}$ and $\sin\theta=\frac{b}{r}$. $(1+i)^{50}=[\sqrt{2}e^{i\pi/4}]^{50}=2^{25}e^{i25\pi/2}=2^{25}(\cos 25\pi/2+i\sin 25\pi/2)=2^{25}i=0+(2^{25})i$.

III. Recall that $Re(z)=\frac{1}{2}(z+\overline{z})$ where $\overline{z}$ is the complex conjugate of z and $\overline{(z^n)}=(\overline{z})^n$.
$Re\left((1+i)^{50}\right)=\frac{1}{2}[(1+i)^{50}+\overline{(1+i)^{50}}]=\frac{1}{2}[(1+i)^{50}+(1-i)^{50}]=$
$\frac{1}{2}\{(1+i)^{50}+[i(-i-1)]^{50}\}=\frac{1}{2}[(1+i)^{50}+i^{50}(-1-i)^{50}]=\frac{1}{2}[(1+i)^{50}+i^{50}(1+i)^{50}]=$
$\frac{1}{2}[(1+i)^{50}-(1+i)^{50}]=0$.

9. Two concentric circles have radii of 5 and 3 units. From a point A on the outer circle two segments are drawn tangent to the inner circle at points B and C while intersecting the outer circle at points D and E (see figure). If O is the center of both circles, what is the length of segment OF, that is perpendicular to DE?
(a) 7/5 (b) 7/2 (c) 5/2 (d) 3/2 (e) none of these



Solution:

In the right triangle $\triangle OBA, OB = 3$ and $OA = 5$, so $BA = 4$. Since $\triangle ODA$ is an isosceles triangle, $\angle ODA = \angle OAD$ and so triangles $\triangle OBD$ and $\triangle OBA$ are congruent. Therefore, $BD = 4$ and $AD = 8$. Now, right triangles $\triangle DFA$ and $\triangle OBA$ having the angle $\measuredangle DAF$ in common are similar. Thus $\frac{AB}{AF} = \frac{OA}{AD} \Longrightarrow \frac{4}{AF} = \frac{5}{8} \Longrightarrow AF = \frac{32}{5} \Longrightarrow OF = AF - OA = \frac{32}{5} - 5 = \frac{7}{5}$. The correct answer is (a).

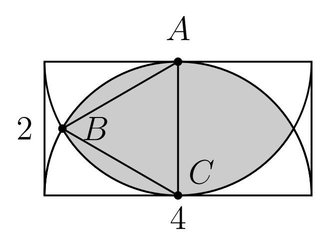
10. How many of the 900 three digit numbers have at least one even digit?
(a) 775 (b) 875 (c) 450 (d) 750 (e) none of these

Solution:

There are 900 three digit numbers and there are five odd digits. Thus, there are $5^3 = 125$ three digit numbers comprised of only odd digits. The other $900 - 125 = 775$ three digit numbers must contain at least one even digit. The correct answer is (a).

11. A rectangle has length 4 and height 2. What is the area of the shaded region, which is the intersection of the two semicircles pictured?
(a) $\dfrac{4\pi}{3} + 2\sqrt{3}$ (b) $\dfrac{4\pi}{3} - 2\sqrt{3}$ (c) $\dfrac{8\pi}{3} - 2\sqrt{3}$ (d) $\dfrac{8\pi}{3} + 2\sqrt{3}$ (e) none of these

Solution:



I. Notice that triangle $\triangle ABC$ is an equilateral triangle of sides 2 and therefore has height $\sqrt{3}$. By symmetry, Area of the shaded region $= 4$(Area of sector ABC) $- 2$(Area of $\triangle ABC$) $= 4\frac{2^2\pi}{6} - 2\frac{1}{2}2\sqrt{3}) = \frac{8\pi}{3} - 2\sqrt{3}$. The correct answer is (c).

II. Place a rectangular coordinate system with the origin at D, the midpoint of the line segment AC. The equation of the concave down circle with center at $(0, -1)$ and radius 2 is $x^2 + (y+1)^2 = 4$. The equation of the portion of this circle above the x-axis is $y = -1 + \sqrt{4 - x^2}$ and it intersects the positive side of x-axis at the point $(\sqrt{3}, 0)$. By symmetry, Area of the shaded region $= 4\int_0^{\sqrt{3}}(-1+\sqrt{4-x^2})dx = 4(-\sqrt{3} + \int_0^{\sqrt{3}} \sqrt{4-x^2}dx)$. Now, evaluate the remaining integral by trigonometric substitution: $x = 2\sin\theta$, $-\pi/2 \le \theta \le \pi/2$ which results in $\sqrt{4-x^2} = \sqrt{4 - 4\sin^2\theta} = 2|\cos\theta| = 2\cos\theta$ and $dx = 2\cos\theta d\theta$. Area of the shaded region $= 4(-\sqrt{3} + \int_0^{\pi/3} 4\cos^2\theta d\theta) = 4[-\sqrt{3} + \int_0^{\pi/3} 2(1+\cos 2\theta)d\theta] = -4\sqrt{3} + 8(\theta + \frac{1}{2}\sin 2\theta)|_0^{\pi/3} = -4\sqrt{3} + 8(\frac{\pi}{3} + \frac{\sqrt{3}}{4}) = \frac{8\pi}{3} - 2\sqrt{3}$.

12. What is the value of

$$\sin^2 1^\circ + \sin^2 2^\circ + \sin^2 3^\circ + \ldots + \sin^2 90^\circ$$

(a) 0 (b) 45 (c) 45.5 (d) 90 (e) none of these

Solution:

Regrouping the terms and using the fact that $\sin\theta^\circ = \cos(90^\circ - \theta^\circ)$ we get $\sin^2 1^\circ + \cdots + \sin^2 90^\circ = (\sin^2 1^\circ + \sin^2 89^\circ) + \cdots + (\sin^2 44^\circ + \sin^2 46^\circ) + (\sin^2 45^\circ + \sin^2 90^\circ) = (\cos^2 89^\circ + \sin^2 89^\circ) + \cdots + (\cos^2 46^\circ + \sin^2 46^\circ) + (\sin^2 45^\circ + \sin^2 90^\circ) = 44 + (\frac{1}{2} + 1) = 45.5$. The correct answer is (c).

13. Given that $\cos x = \tan x$, what is $\sin x$?

(a) $\dfrac{-1+\sqrt{2}}{2}$ (b) $\dfrac{-1+\sqrt{3}}{2}$ (c) $\dfrac{-1+\sqrt{5}}{2}$ (d) $\dfrac{-2+\sqrt{5}}{2}$ (e) none of these

Solution:

Since $\cos x = \tan x = \frac{\sin x}{\cos x}$, the angle x is in the first or second quadrant and $\cos^2 x = \sin x$. Thus $\sin x$ is nonnegative, and substituting $\cos^2 x = 1 - \sin^2 x$ in the second equation results in $\sin^2 x + \sin x - 1 = 0$. The only nonnegative solution of this equation is $\sin x = \frac{-1+\sqrt{5}}{2}$. The correct answer is (c).

14. Read the following 5 statements carefully:

(i) Statement (ii) is true.

(ii) At most one of these 5 statements is true.

(iii) All 5 of these statements are true.

(iv)

(v)

The last two statements are printed in invisible ink. Which of the statements are true?

(a) only (i) (b) only (iv) and (v) (c) all of them

(d) none of them (e) cannot be determined

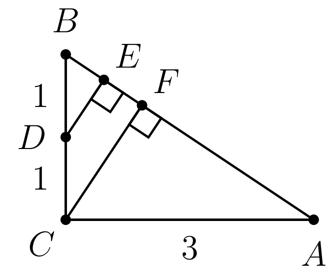
Solution:

If the first statement (i) were true, it would make (ii) true, but then we would have two true statements already, and that contradicts (ii). Therefore, (i) must be false, and so is (ii). Clearly (iii) is false, and the only way to have at least two correct statements (remember, (ii) is false) is to have (iv) and (v) both true. The correct answer is (b).

15. In the right triangle ABC, leg AC is fifty percent longer than leg BC. Let D be the midpoint of side $\overline{BC}$, and $\overline{DE}$ be perpendicular to $\overline{AB}$. What is the ratio of the area of triangle DBE to the area of triangle ABC?

(a) $1/13$ (b) $1/10$ (c) $1/\sqrt{13}$ (d) $1/16$ (e) none of these

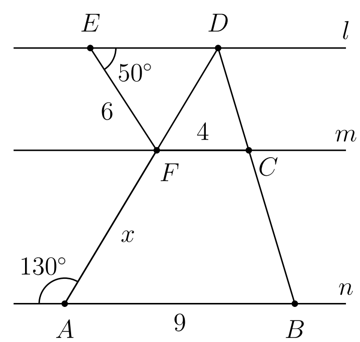
Solution:



I. We can assume that side $BC = 2$ and then $AC = 3$, and $AB = \sqrt{13}$. Using the notations of the figure, $AC \cdot BC = AB \cdot CF$, from which $CF = \frac{6}{\sqrt{13}}$. Since DE is the midsegment of triangle CFB, $DE = \frac{1}{2}CF = \frac{3}{\sqrt{13}}$. Another application of the Pythagorean theorem yields $BE = \frac{2}{\sqrt{13}}$. Thus the area of $\triangle DBE$ is $\frac{1}{2}\frac{2}{\sqrt{13}}\frac{3}{\sqrt{13}} = \frac{3}{13}$.
Since the area of $\triangle ABC = 3$, the ratio of the areas is $1/13$. The correct answer is (a).

II. The second solution uses less calculations. Notice that $\triangle DBE \sim \triangle CBF \sim \triangle ACF$ by finding congruent corresponding angles. The ratio of the areas of two similar triangles is the square of the ratio of similarity. Thus $\text{Area}(\triangle CBF) = 2^2 \cdot \text{Area}(\triangle DBE)$, and $\text{Area}(\triangle ACF) = 3^2 \cdot \text{Area}(\triangle DBE)$. Since these two triangles form $\triangle ABC$, $\text{Area}(\triangle ABC) = (4+9) \cdot \text{Area}(\triangle DBE)$, the ratio is 1:13.

16. In the figure at the right lines l, m and n are all parallel. Which one of the following is closest to the value of $x = AF$?



(a) 6.75 (b) 7 (c) 7.25 (d) 7.50 (e) 7.75

Solution:

Using the Alternate Interior Angle theorem $\angle EDA = 50°$, so $\triangle FED$ is an isosceles triangle and $DF = 6$. By the similarity of triangles $\triangle DFC$ and $\triangle DAB$, due to the fact that $m \| n$, $\frac{DF}{DA} = \frac{FC}{AB}$ and so $\frac{6}{6+x} = \frac{4}{9} \Longrightarrow x = 7.5$. The correct answer is (d).

17. Miss Black, Mr. Crimson, Mrs. Gold, Mr. Green, and Mr. White each own a car that has a color that is the name of one of the other four. Mr. Green's sister is married to the owner of the crimson car. The husband of the owner of the white car carpools with the owner of the green car, who is engaged to Miss Black. Who owns the black car?

(a) Miss Black (b) Mr. Crimson (c) Mrs. Gold (d) Mr. Green (e) Mr. White

Solution:

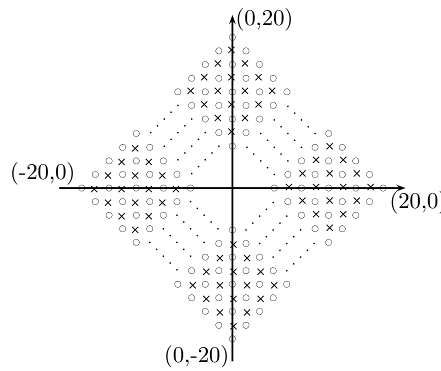
From the third sentence, the owner of the white car has a husband, and the only married woman in the company is Mrs. Gold, she owns the white car. Mr. Green's sister is married to the owner of the crimson car, and that can't be Mr. Crimson or Mr. Green, the only remaining man is Mr. White, he owns the crimson car. Miss Black's car is not black (by the first rule, nobody owns a car of the same color as their name), not crimson or white (they already have owners), and not green (its owner is a man). Miss black owns the gold car. What color is Mr. Green's car? The gold, white and crimson cars have other owners, he must own the black car. The correct answer is (d).

18. How many pairs of integers x and y will satisfy $|x| + |y| \le 20$?

(a) 210 (b) 420 (c) 820 (d) 841 (e) none of these

Solution:

I. Let us start with $|x| = 20$. Then y must be zero so we have $(20, 0)$ and $(-20, 0)$ as our two possible options. If $|x| = 19$, then $|y|$ can be either one or zero. Thus, there are six options of ordered pairs - $(-19, 0), (-19, 1), (-19, -1), (19, 0), (19, 1), (19, -1)$ - since x can be either ± 19, and y can be ± 1 or 0. In general, then, if n is an integer, $1 \le n \le 20$ and $|x| = n$, then x must be either $\pm n$, and there are $2(20 - n) + 1$ possibilities for y, or $4(20 - n) + 2$ possible ordered pairs. Thus, if $1 \le |x| \le 20$, we have $\sum_{n=1}^{20} 4(20 - n) + 2$ possible ordered pairs (x, y). This is an arithmetic series, so $\sum_{n=1}^{20} 4(20 - n) + 2 = \frac{20}{2}(78 + 2) = 800$. Finally, if $|x| = 0$, then $x = 0$, and y can be $\pm 20, \pm 19, \pm 18, ..., \pm 1$ or 0 . This gives us another 41 ordered pairs for a total of $800 + 41$. The correct answer is (d).



II. A more visual solution is to graph the points with coordinates (x, y) such that $|x| + |y| \le 20$ (see figure). There are many ways to count them, perhaps the trickiest is to count the points where the sum of the coordinates is even (those marked with $\circ$), there are 21×21 of them, and then those where the sum of the coordinates is odd (marked with x), there are 20×20 of those. We have a total of $21^2 + 20^2 = 841$ points that satisfy the condition.

19. Players A and B alternate tossing a biased coin, with A going first. A wins the coin if A tosses a tail before B tosses a head; otherwise B wins. If the probability of a head is p, what should be the value of p to make this game fair to both players?

(a) $1/2$ (b) $1/3$ (c) $\sqrt{2}/2$ (d) $\sqrt{3} - 1$ (e) none of these

Solution:

I. The probability that player A wins in the first round is $1 - p$, that he wins in the second round of flips is $[(p)(1 - p)](1 - p)$, and, in general, that he wins in the n^{th} round is $[(p)(1 - p)]^n(1 - p)$. We need $\sum_{n=1}^{\infty}[(p)(1 - p)]^n(1 - p) = \frac{1}{2}$. Summing this geometric series, we get $\frac{1}{2} = \frac{1-p}{1-p(1-p)}$, so $p^2 + p - 1 = 0$ and $p = \frac{1}{2}(-1 + \sqrt{5})$. This is none of the given options, so the correct answer is (e).

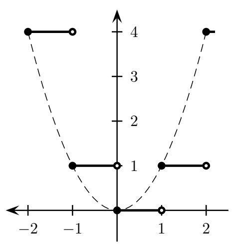
II. We can make use of the fact that the probability of A winning is still $1/2$ after both players flipped the coin once and nobody won. More formally;

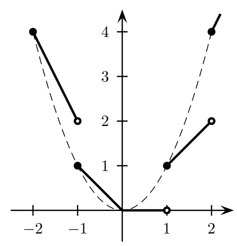
$P(\text{A wins}) = P(\text{A wins in first round}) + P(\text{nobody wins in first round and A wins later})$.

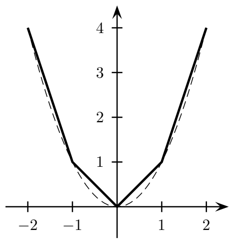
$\frac{1}{2} = (1 - p) + p(1 - p)\frac{1}{2}$, which leads to the same quadratic equation as before.

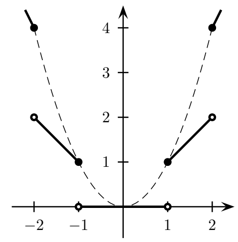
Note: The answer is the reciprocal of the famous "golden ratio", and comes up in many seemingly unrelated problems.

20. Which of the graphs is the graph of the function $f(x) = x \cdot [x]$, where $[x]$ denotes the largest integer less than or equal to x.

(a) 

(b) 

(c) 

(d) 

(e) none of these

Solution:

At every integer $f(x) = x^2$, so those points fit the parabola $y = x^2$. Between two consecutive integers n and $n+1$ the function $f(x)$ is linear, with a slope of n. The only graph matching these requirements is answer (b).

21. Find a and b so that the graph of $y = x + k$ intersects the graph of $(x-3)^2 + (y+2)^2 = 50$ in one or more points if and only if $a \leq k \leq b$.

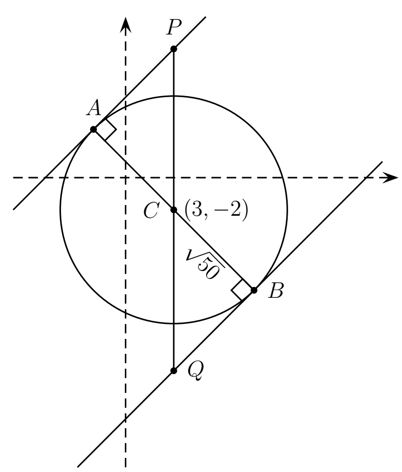
(a) $a = -13$, $b = 7$ (b) $a = -15$, $b = 10$ (c) $a = -13$, $b = 5$

(d) $a = -15$, $b = 5$ (e) none of these

Solution:

I. Substitute $y = x + k$ into the other equation: $(x-3)^2 + ((x+k)+2)^2 = 50$, and after rearranging this we can get to $x^2 + (k-1)x + ((k+2)^2 - 41)/2 = 0$. For this quadratic equation to have real solutions the discriminant must be non-negative: $(k-1)^2 - 2((k+2)^2 - 41) \geq 0$ which simplifies to $-(k+15)(k-5) \geq 0$, from which $-15 \leq k \leq 5$, so $a = -15$, $b = 5$. The correct answer is (d).

II. A more geometric solution is to graph the circle with center $C = (3, -2)$ and radius $\sqrt{50}$. We are looking for the y-intercepts of the two tangent lines with slope 1. Let the two tangent lines intersect the circle at A and B, and the line $x = 3$ at P and Q (see figure). Both $\triangle CPA$ and $\triangle CQB$ are right isosceles with a leg of $\sqrt{50}$, thus have a hypothenuse of 10. Then $P = (3, 8)$, $Q = (3, -12)$ and the corresponding y-intercepts are -15 and 5.



22. What is the sum of the y-components of the real ordered pair solutions (x, y) of the system of equations

$$\begin{aligned} 4^{xy+4} &= 8^{x^2-y-1} \\ y &= x+1 \ ? \end{aligned}$$

(a) 5 (b) 6 (c) 7 (d) 8 (e) none of these

Solution:

Substitute $y = x+1$ into the first equation: $4^{x(x+1)+4} = 8^{x^2-(x+1)-1}$. Multiply the exponents and writing both sides as powers of 2 we get: $2^{2x^2+2x+8} = 2^{3x^2-3x-6}$. Since the exponential function is one-to-one, this is equivalent with
$2x^2+2x+8 = 3x^2-3x-6$, or $x^2-5x-14=0$. The solutions for x are 7 and -2, the corresponding y values are 8 and -1, their sum is 7, the correct answer is (c).

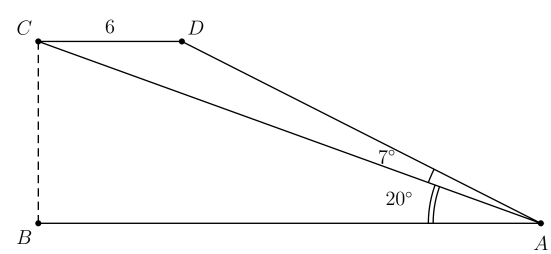
23. An observer spots an airplane that is flying horizontally on a course that will take it directly overhead in a few minutes. The plane is known to cruise at 540 miles per hour. On the initial sighting the angle of elevation was 20°, and 40 seconds later it was 27°. What is the altitude of the plane, in miles?

(a) $\dfrac{360 \sin 153^\circ \sin 20^\circ}{\sin 7^\circ}$ (b) $\dfrac{360 \sin 153^\circ \sin 20^\circ}{\sin 27^\circ - \sin 20^\circ}$

(c) $\dfrac{6 \sin 153^\circ \sin 20^\circ}{\sin 27^\circ - \sin 20^\circ}$ (d) $\dfrac{6 \sin 153^\circ \sin 20^\circ}{\sin 7^\circ}$ (e) none of these

Solution:

Using the labeling of the picture we can first calculate CD, the distance the plane travels in 40 seconds is $540 \times 40/3600 = 6$ miles. Since $\overline{CD} || \overline{AB}$, $m\angle CDA = 180^\circ - m\angle BAD = 153^\circ$. Using the Law of Sines in $\triangle ADC$, $\dfrac{6}{\sin 7^\circ} = \dfrac{AC}{\sin 153^\circ}$.



Using the value derived from this for AC in the right triangle $\triangle ABC$:

$BC = \sin 20^\circ AC = \dfrac{6 \sin 153^\circ \sin 20^\circ}{\sin 7^\circ}$. The correct answer is (d).

Note: There are lots of other ways to express the height, many involving other trigonometric functions, where further trigonometric identities may be necessary to find the correct option given.

24. Which of the following six statements are true about the cubic polynomial $P(x) = 2x^3 + x^2 + 3x - 2$?

(i) It has exactly one positive real root.

(ii) It has either one or three negative roots.

(iii) It has a root between 0 and 1.

(iv) It must have exactly two real roots.

(v) It has a negative root between -2 and -1.

(vi) It has no complex roots.

(a) only (i) (b) only (i), (iii) and (vi) (c) only (ii), (iii) and (iv)

(d) only (i) and (iii) (e) only (iii), (iv) and (v)

Solution:

First we notice that the function is increasing by checking the derivative $P'(x) = 6x^2 + 2x + 3 > 0$. That means $P(x)$ will have exactly one real root (and two complex roots). Since $P(0) = -2$ and $P(1) = 4$ the root is between 0 and 1 by the Intermediate Value Theorem. Thus only statements (i) and (iii) are true, the correct answer is (d).

Note: One can also solve the problem using Descartes' rule of signs and checking the function values in different intervals.

25. Find the largest positive integer that has the property that if you divide it into each of 4201, 6301, and 7351 the remainder will be 1.

(a) 1050 (b) 2100 (c) 50 (d) 150 (e) none of these

Solution:

We seek the largest common divisor of 4200, 6300 and 7350. Factoring each number into product of primes: $4200 = 2^3 \cdot 3 \cdot 5^2 \cdot 7$, $6300 = 2^2 \cdot 3^2 \cdot 5^2 \cdot 7$, $7350 = 2 \cdot 3 \cdot 5^2 \cdot 7^2$ we can see that the largest common divisor is $2 \cdot 3 \cdot 5^2 \cdot 7 = 1050$. The correct answer is (a).

26. A triangle has one vertex at $(0, 0)$ and the other two on the graph of $y = -2x^2 + 54$, at (x, y) and $(-x, y)$ where $0 < x < \sqrt{27}$. Find x such that the corresponding triangle has maximum area.

(a) $\sqrt{27}/2$ (b) 3 (c) $\sqrt{3}$ (d) $2\sqrt{3}$ (e) none of these

Solution:

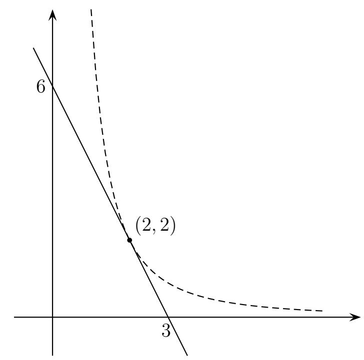
The area of the triangle is $A(x) = \frac{1}{2}(2x)y = x(-2x^2 + 54) = 54x - 2x^3$; where $0 < x < \sqrt{27}$. For the function $A(x)$ to have a maximum $A'(x) = 54 - 6x^2 = 0$ which holds if and only if $x = \pm 3$. Since $x = -3$ is outside the domain and $A'' = -12x < 0$ in the domain, A achieves its absolute maximum at $x = 3$. The correct answer is (b).

27. The line $\mathcal{L}$ is tangent to the curve $x^2y = 8$ at the point $(2, 2)$. Compute the area below $\mathcal{L}$ in the first quadrant.

(a) 9 (b) 8 (c) 1 (d) $4\sqrt[3]{2}$ (e) $4\sqrt{2}$

Solution:

The slope of the tangent line will be equal to the derivative of the function $y = 8/x^2$ at point $(2, 2)$.
$m_{\text{tangent}} = \frac{d}{dx}(8x^{-2})|_{x=2} = -16x^{-3}|_{x=2} = -2$. The tangent line $\mathcal{L}$ through point $(2, 2)$ with the slope -2 will cross the axes at $(0, 6)$ and $(3, 0)$. The region below the line $\mathcal{L}$ in the first quadrant is a right triangle with legs of length 6 and 3. The desired area is $\frac{1}{2} \times 6 \times 3 = 9$.
Note: The intercepts of the tangent line can also be found by first finding the equation of the tangent line.



28. Which of the following is always true about a function $f(x)$ on the interval $[a, b] = \{x \mid a \leq x \leq b\}$?

(a) If $f(x) \geq 0$ on $[a, b]$, then $\displaystyle\int_a^b f(x)dx \leq \int_a^b f^2(x)dx$

(b) If $f(x)$ is increasing on $[a, b]$, then $f^2(x)$ is increasing on $[a, b]$

(c) If $f(x)$ is increasing on $[a, b]$, then $f(x) \geq 0$ on (a, b)

(d) If $f(x)$ attains a minimum at c where $a < c < b$, then $f'(c) = 0$

(e) none of these

Solution:

(a) may be false since if $0 < f(x) < 1$, $f^2(x) < f(x)$ and $\int_a^b f(x)dx > \int_a^b f^2(x)dx$.

(b) may be false since if $f(x) < 0$, $\frac{d}{dx}f^2(x) = 2f(x)f'(x) < 0$ when $f'(x) > 0$ and so $f^2(x)$ is decreasing while $f(x)$ is increasing.

(c) may be false since a function can be negative and increasing.

(d) may be false since a function may not be differentiable at $x = c$ for which it attains its minimum.

Since none of the statements are always true, the correct answer is (e).

29. An equation for the line that both passes through (10,-1) and is perpendicular to $y = \frac{1}{4}x^2 - 2$ is:

(a) $4x + y = 39$ (b) $2x + y = 19$ (c) $x + y = 9$ (d) $x + 2y = 8$ (e) none of these

Solution:

Suppose the perpendicular line intersects the curve at the point (a, b). The slope of this line m is the negative reciprocal of the slope of the tangent line: $m = -\frac{1}{y'(a)} = -\frac{1}{\frac{x}{2}|_{x=a}} = -\frac{2}{a}$. The point (a, b) is both on the perpendicular line $y + 1 = -\frac{2}{a}(x - 10)$ and the curve. Reducing the nonlinear system $b + 1 = -\frac{2}{a}(a - 10)$, $b = \frac{1}{4}a^2 - 2$ to one equation in one variable results in $a^3 + 4a - 80 = 0$. Using the Rational Zero Theorem (or an easy guess), we see that one solution of this equation is $a = 4$ and it factors as $(a - 4)(a^2 + 4a + 20) = 0$. Since the equation $a^2 + 4a + 20 = 0$ has no real zeros, then the only value for a is 4. The equation of the perpendicular line is $y + 1 = -\frac{2}{4}(x - 10)$ which simplifies to $x + 2y = 8$. The correct answer is (d).

30. Figure $ABCDE$ is a regular pentagon with all sides equal to 4. Which of the following is (are) a correct solution(s) for the length of AC?

(i) $2\csc(18°)$ (ii) $2\sec(72°)$ (iii) $\sqrt{32 - 32\cos(108°)}$

(a) only (i) is correct (b) only (ii) is correct (c) only (iii) is correct

(d) only (i) and (ii) are correct (e) all are correct

Solution:

Denote the center of the pentagon by O. Because the pentagon is regular $\angle AOB = \frac{360°}{5} = 72°$. Triangle AOB is isosceles, so each base angle is 54°. Applying the law of cosines to $\triangle ABC$ yields $AC = \sqrt{32 - 32\cos 108°}$.

Let the point F bisect AE, so $AF = 2$. By symmetry of the regular pentagon, the perpendicular bisector of AE from F passes through O and C. Angle $AOF = 72°/2 = 36°$ and is exterior to the isosceles triangle AOC and so $\angle ACF = 36°/2 = 18°$ and so $\angle FAC = 90° - 18° = 72°$. Looking at the right triangle $\triangle AFC$ yields $AC = 2\sec 72° = 2\csc 18°$.

All the options are correct, the correct answer is (e).