2005 STATE MATH CONTEST

GRADE 7-9 PRETEST SOLUTIONS

1. If the length of a rectangle were increased by 20% and the width by 25%, by what percent would its area increase?

   (a) 22.5% (b) 45% (c) 35% (d)* 50% (e) 65%

   Solution:

   The area of a rectangle is length times width ($l \times w$). If the length is increased by 20% and the width is increased by 25%, the new area would be (1.2 × length) × (1.25 × width) $= 1.2 \times 1.25 \times l \times w = 1.5 \times l \times w$. This is a 50% increase in the area. The correct answer is (d).

2. Billy has three times as many nickels as dimes. If he had 45 more dimes he would have twice as much money. How many nickels and dimes does he have altogether now?

   (a) 18 (b) 52 (c) 54 (d)* 72 (e) none of these

   Solution:

   I. The first solution uses some algebra. Call the number of dimes Billy has d. Then he has $3d$ nickels, and $.10d + .05(3d)$ dollars. This is equal to the value of 45 dimes (adding those would double his money), thus $.10d + .05(3d) = .10 \cdot 45$. Solving this simple equation leads to $d = 18$. Billy has 18 dimes and three times as many, or 54 nickels. He has a total of 72 coins, the correct answer is d.

   II. The second solution uses no algebra at all. First we notice that adding 45 dimes (a value of \$4.50) doubles Billy's money, so he must have \$4.50. Since he has three times as many nickels as dimes, we can group three nickels and a dime together as many times as possible, with no remainder. Each of these groups have four coins and a value of \$.25, so we must have made $4.50/.25 = 18$ groups. The number of coins must then be $4 \times 18 = 72$, which is answer (d).

3. The faces of eight congruent cubes are painted. Every face is colored with one color, but different faces of the cubes can be colored differently. What is the largest number of colors we could have used if we can join the colored cubes together to form a big cube that is of the same color from the outside?

   (a) 13 (b)* 25 (c) 30 (d) 48 (e) none of these

   Solution:

   When the cubes are put together, each cube has three faces on the inside of the larger cube, and three faces on the outside of the cube. Each inside face can have a different color, that gives us $3 \times 8 = 24$ colors. Since the outside of the big cube has to have one color this gives us the 25th color. The correct answer is (b).

4. A rectangular bin having a bottom but no top is to be constructed with a length that is twice the width and with a capacity of 160ft^3. The material for the sides will cost $\$3/\text{ft}^2$ and that for the bottom $\$4/\text{ft}^2$. Find a formula for the cost of materials for that box in terms of the width w, where w is in feet.

   (a) $16w^2 + \dfrac{1440}{w}$ (b) $16w^2 + \dfrac{1960}{w}$ (c) $16w^2 + \dfrac{960}{w}$ (d) $8w^2 + \dfrac{1960}{w}$ (e)* $8w^2 + \dfrac{1440}{w}$

   Solution:

   Using standard notation for width (w) and height (h), the volume of the box is $w(2w)h = 160$ or $h = \dfrac{160}{2w^2} = \dfrac{80}{w^2}$. The price of the material for the box is then $4(w \times 2w) + 2 \times 3(h \times 2w) + 2 \times 3(h \times w) = 8w^2 + 18(h \times w) = 8w^2 + \frac{1440}{w}$. The correct answer is (e).

5. The product of two consecutive even integers is 34 more than the sum of those integers. Which number can be one of the two integers?

   (a) -8 (b)* -4 (c) 0 (d) 4 (e) 10

   Solution:

   One can guess the solution quickly after realizing that the two integers must even (to make the product and the sum have the same parity), but here is a quick algebraic solution:

   Denote the two numbers by n and $n + 2$, then $n(n + 2) = n + (n + 2) + 34$. This equation simplifies to $n^2 = 36$, from which $n = \pm 6$. Correspondingly, the other numbers $n + 2 = 8$ and -4, respectively. Of these four numbers only -4 is listed, the correct answer is (b).

6. The rate at which a river flows is one third the speed of a boat in still water. If that boat travels down that river for two hours and then back upriver for two hours, it will be 16 miles short of its starting point. What is the speed of the boat in still river?

(a) 4mph (b) 8mph (c)* 12mph (d) 24mph (e) none of these

Solution:

I. First, a "tricky" solution. Drop a ball in the water at the beginning of the trip. Compared to the water, the boat travels up and down stream with the same speed. Therefore it will arrive back to the ball after 4 hours. Since the ball travelled 16 miles in 4 hours, the speed of the river must be 4mph and the speed of the boat is three times as much, 12mph. The correct answer is (c).

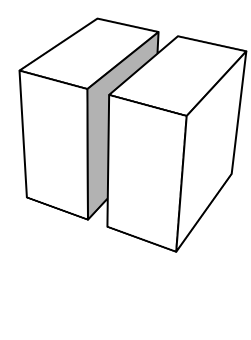
II. A more algebraic solution: denote the speed of the river by r, and the speed of the boat in still water by $3r$. Then the boat must travel at the speed of $3r + r = 4r$ downstream, and $3r - r = 2r$ upstream. In two hours it went $2 \times 4r = 8r$ downstream and $2 \times 2r = 4r$ upstream. The difference $8r - 4r = 4r = 16$ miles. From this $r = 4$mph and therefore the speed of the boat is $3r = 12$mph.

7. A cube is sliced up by planes parallel with its faces. How many planes do we use to slice if the total of the surface areas of the resulting pieces is twice the surface area of the original cube?

(a) 2 (b)* 3 (c) 6 (d) 12 (e) none of these

Solution:

Every slice increases the total surface area by the area of two sides, since two new squares come to the surface along the cut exactly congruent with the original sides. Since the original surface was six of these squares we will need 3 cuts to double it. The correct answer is (b).

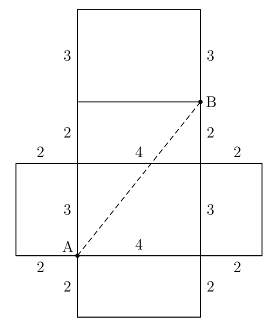


Note that the answer does not depend on the location of the cuts; they may be parallel with one another or may have some cross others.

8. The edges of a rectangular box are 2, 3, and 4. What is the length of the shortest path from one vertex to the opposite vertex on the surface of the box?

(a) $\sqrt{29}$ (b)* $\sqrt{41}$ (c) $\sqrt{45}$ (d) $\sqrt{53}$ (e) none of these

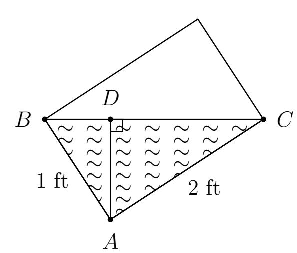
Solution:
The best way to visualize the problem is to unfold the box and draw its net in two dimensions (see picture). The straight line connecting points A and B is the shortest path connecting opposite vertices. Using the Pythagorean theorem $\sqrt{4^2+5^2} = \sqrt{41}$, is the length of the shortest path. Notice that one may need to unfold the net three different ways to find the actual shortest path represented by a straight line segment. The correct answer is (b).



9. A rectangular aquarium that is 6 feet long, 2 feet wide and 1 foot high is half-full of water. It is slowly tilted along one of the 6 foot edges until the water is just ready to spill out. How deep is the water at its deepest point when the water just starts to spill?

(a)* $\dfrac{2}{\sqrt{5}}$ (b) $\sqrt{52}$ (c) $\dfrac{1}{\sqrt{5}}$ (d) $\sqrt{5}$ (e) none of these

Solution:
The picture shows the shortest side of the aquarium at the moment water starts spilling. Since it was half full, at this time the water level will connect vertices B and C. The depth of the water is given by the AD, the altitude corresponding to the hypothenuse of the right triangle ABC. Using Pythagorean theorem the length of BC is $\sqrt{5}$. Since the angles are the same, $\triangle ABC \sim \triangle DBA$. Therefore, $AD/AC = AB/BC$, or $AD/1 = 2/\sqrt{5}$. The correct answer is (a).



10. Peter, Quincy, Robert, Shawn, and Todd play the following game. Peter goes out of the room, and the other four decide on who will hide a ring. Two kids will tell three statements each to the entering Peter, of which two will be true, one will be a lie. Peter heard the following when he returned:

Quincy:	"I don't have the ring."
	"Robert has the ring."
	"I have played this game many times."
Robert:	"I don't have the ring."
	"Quincy lied when he said that I have the ring."
	"Shawn does not have the ring either."

Who is holding the ring?

(a) Quincy (b) Robert (c)* Shawn (d) Todd (e) cannot be determined

Solution:

Look at Quincy's responses first. If his first statement were a lie, than he would have the ring but his second statement would then also have to be a lie. Since he can have only one lie, this can't be. If Quincy's third statement were a lie, then the second statement would be true and Robert would have the ring. But looking at Robert's statements, if he had the ring, then his first two statements would be lies, which is not allowed. So Quincy's second statement must be a lie. That means Robert doesn't have the ring. These also mean that Robert's first two statements must be the truth (he doesn't have the ring and the Quincy's second statement is a lie) so Robert's third statement must be a lie. This means that Shawn must have the ring. The correct answer is (c).

11. Four friends; Alex, Bob, Carl, and Daniel would like to climb through a dark and narrow tunnel. They only have one lamp (necessary to be able to get through), and only two of the children can get in the tunnel together. Alex could get through in 1 minute, Bob in 2 minutes, Carl in 5 and Daniel in 6 minutes. At least how many minutes must the battery hold in their lamp for all of them to get through?

(a) 8 (b) 10 (c) 12 (d) 15 (e)* none of these

Solution:

To minimize the time travelled we must make sure that the people with the longest travel times go across together and only once. One way to do this is to have Alex and Bob go across first and have Bob come back with the lamp. Then Carl and Daniel (the slow pokes) go across together and have Alex bring back the lamp. Then Alex and Bob will go across together. The total travel time would then be $2 + 2 + 6 + 1 + 2 = 13$ minutes. This is the minimum amount of time necessary to get through. Since this is not one of the options, the correct answer is (e).

12. Which of the following statements are true?

(i) The sum of two rational numbers must be rational.

(ii) The sum of two irrational numbers must be irrational.

(iii) The product of two rational numbers must be rational.

(iv) The product of two irrational numbers must be irrational.

(a)* only (i) and (iii) (b) only (i) and (ii) (c) only (i), (ii) and (iii)

(d) all of them (e) none of them

Solution:

(i) and (iii) are true, the rational numbers are closed under the operations of addition and multiplication. But not the irrational numbers. For example, $\sqrt{2} + (3 - \sqrt{2}) = 3$ and $\sqrt{5} \times \sqrt{5} = 5$ shows that the other two statements are false. The correct answer is (a).

13. What is the sum of the digits of the smallest positive integer that gives a remainder of 4 when divided by 5, a remainder of 5 when divided by 6, and a remainder of 6 when divided by 7?

(a) 3 (b) 7 (c)* 11 (d) 15 (e) none of these

Solution:

If we add one to this mysterious number it will be divisible by 5, 6 and 7. The smallest positive integer like this is the least common multiple of 5, 6 and 7, which is $5 \times 6 \times 7 = 210$. Our number is one less than that, 209, and the sum of its digits is 11. The correct answer is (c).

14. What is the probability that out of three friends, exactly two have the same birth month? (assume that the 12 birth months have equal probabilities).

(a) 11/144 (b) 11/24 (c) 11/96 (d)* 11/48 (e) none of these

Solution:

I. For the first two out of three to have the same birth month, the first friend can have any month (probability 12/12) a second friend must have the same birth month (probability 1/12) and a third must have a different birth month from the other two (probability 11/12). To find the overall probability we multiply these probabilities together and get $1/12 \times 11/12 = 11/144$. Similarly to friends 1 & 2; 1 & 3, or 2 & 3 can have the same birth month. So we must multiply this result by 3 to get the final probability of 11/48. The correct answer is (d).

II. Another way to solve the problem is to look at the complement of the event of exactly two friends having the same birth month. It means either all three having the same birth month (the probability of which is $12/12 \times 1/12 \times 1/12 = 1/144$), or all of them have different birth months (with probability of $12/12 \times 11/12 \times 10/12 = 110/144$). The probability of exactly two of them to have the same birth month is then $1 - 1/144 - 110/144 = 33/144 = 11/48$. The correct answer is (d).

15. For any set S let $n(S)$ denote the number of elements in set S. If $n(A) = 80$, $n(B) = 60$, $n(C) = 90$, what is the maximum number of elements the set $(A \cup B) \cup (B \cap C)$ can have?

(a) 80 (b)* 140 (c) 170 (d) 200 (e) 230

Solution:

To maximize $A \cup B$, they must be disjoint, and then $n(A \cup B) = 140$. Since $B \cap C$ is a subset of B, it will not give any more elements to the union that already contains B, therefore maximum number of elements in the required set is 140, the correct answer is (b).