2005 STATE MATH CONTEST

GRADES 10 – 12

1. What is the sum of the solutions of the equation $\sqrt{4x+15} - 3 = \sqrt{2x-1}$?

   (a) 11 (b) 3 (c) 9 (d) 12 (e) 5

2. The planes containing the faces of a cube divide the space into several regions. How many regions?

   (a) 6 (b) 12 (c) 24 (d) 27 (e) 36

3. How many digits (in base 10) do 2^{2005} and 5^{2005} have altogether?

   (a) 2006 (b) 2005 (c) 4010 (d) 2004 (e) none of these

4. A convoy is traveling at 30 miles per hour. A messenger on a motorcycle traveling at 50 miles per hour can go from one end of the convoy to the other and back again in 23 minutes. About how many miles long is the convoy?

   (a) 8.2 (b) 5.6 (c) 7.1 (d) 6.5 (e) 6.1

5. In a drawer there are 18 socks, 5 black, 6 white and 7 green. If one morning one pulls out two socks at random (with closed eyes), what is the probability of obtaining a matching pair?

   (a) 46/153 (b) 13/36 (c) 4/105 (d) 55/162 (e) none of these

6. The negation of the statement "No hungry man is happy" is:

   (a) All hungry man are happy (b) All hungry man are unhappy (c) At least one hungry man is happy (d) No hungry man is unhappy (e) At least one hungry man is unhappy

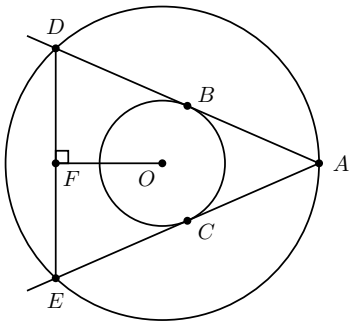
7. If $x-1$, x^2-1, x^3-1, and x^4-1 all are factors of a polynomial $p(x)$, what is the smallest possible degree of $p(x)$?

(a) 5 (b) 6 (c) 7 (d) 10 (e) 12

8. What is the real part of $(1+i)^{50}$?

(a) 0 (b) 2^{25} (c) -2^{25} (d) 2^{50} (e) -2^{50}

9. Two concentric circles have radii of 5 and 3 units. From a point A on the outer circle two segments are drawn tangent to the inner circle at points B and C while intersecting the outer circle at points D and E (see figure). If O is the center of both circles, what is the length of segment OF, that is perpendicular to DE?

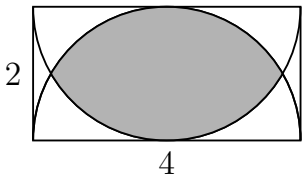


(a) 7/5 (b) 7/2 (c) 5/2 (d) 3/2 (e) none of these

10. How many of the 900 three digit numbers have at least one even digit?

(a) 775 (b) 875 (c) 450 (d) 750 (e) none of these

11. A rectangle has length 4 and height 2. What is the area of the shaded region, which is the intersection of the two semicircles pictured?



(a) $\frac{4\pi}{3}+2\sqrt{3}$ (b) $\frac{4\pi}{3}-2\sqrt{3}$ (c) $\frac{8\pi}{3}-2\sqrt{3}$
(d) $\frac{8\pi}{3}+2\sqrt{3}$ (e) none of these

12. What is the value of

$$\sin^2 1^\circ + \sin^2 2^\circ + \sin^2 3^\circ + \ldots + \sin^2 90^\circ$$

(a) 0 (b) 45 (c) 45.5 (d) 90 (e) none of these

13. Given that $\cos x = \tan x$, what is $\sin x$?

(a) $\dfrac{-1+\sqrt{2}}{2}$ (b) $\dfrac{-1+\sqrt{3}}{2}$ (c) $\dfrac{-1+\sqrt{5}}{2}$ (d) $\dfrac{-2+\sqrt{5}}{2}$ (e) none of these

14. Read the following 5 statements carefully:

(i) Statement (ii) is true.

(ii) At most one of these 5 statements is true.

(iii) All 5 of these statements are true.

(iv)

(v)

The last two statements are printed in invisible ink. Which of the statements are true?

(a) only (i) (b) only (iv) and (v) (c) all of them

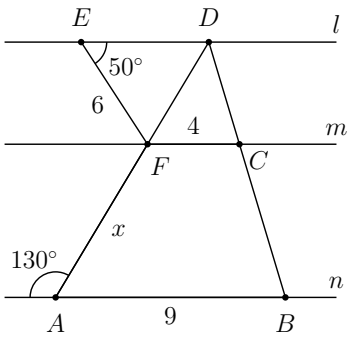
(d) none of them (e) cannot be determined

15. In the right triangle ABC, leg AC is fifty percent longer than leg BC. Let D be the midpoint of side $\overline{BC}$, and $\overline{DE}$ be perpendicular to $\overline{AB}$. What is the ratio of the area of triangle DBE to the area of triangle ABC?

(a) 1/13 (b) 1/10 (c) $1/\sqrt{13}$ (d) 1/16 (e) none of these

16. In the figure at the right lines l, m and n are all parallel. Which one of the following is closest to the value of $x = AF$?

(a) 6.75 (b) 7 (c) 7.25 (d) 7.50 (e) 7.75



17. Miss Black, Mr. Crimson, Mrs. Gold, Mr. Green, and Mr. White each own a car that has a color that is the name of one of the other four. Mr. Green's sister is married to the owner of the crimson car. The husband of the owner of the white car carpools with the owner of the green car, who in turn is engaged to Miss Black. Who owns the black car?

(a) Miss Black (b) Mr. Crimson (c) Mrs. Gold (d) Mr. Green (e) Mr. White

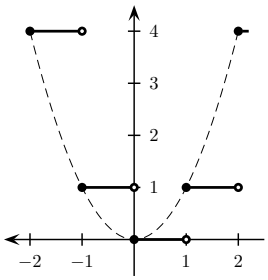
18. How many pairs of integers x and y will satisfy $|x| + |y| \leq 20$?

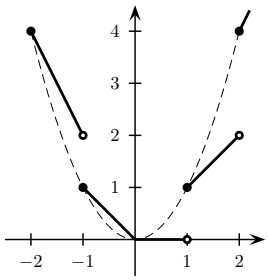
(a) 210 (b) 420 (c) 820 (d) 841 (e) none of these

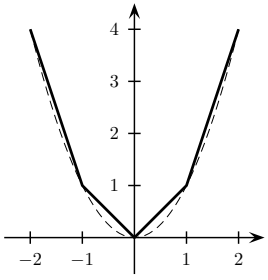
19. Players A and B alternate tossing a biased coin, with A going first. A wins the coin if A tosses a tail before B tosses a head; otherwise B wins. If the probability of a head is p, what should be the value of p to make this game fair to both players?

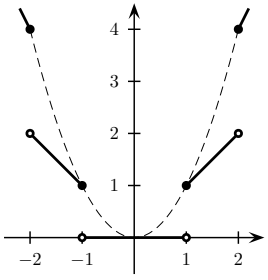
(a) $1/2$ (b) $1/3$ (c) $\sqrt{2}/2$ (d) $\sqrt{3} - 1$ (e) none of these

20. Which of the graphs is the graph of the function $f(x) = x \cdot [x]$, where $[x]$ denotes the largest integer less than or equal to x.

(a) 

(b) 

(c) 

(d) 

(e) none of these

21. Find a and b so that the graph of $y = x+k$ intersects the graph of $(x-3)^2+(y+2)^2 = 50$ in one or more points if and only if $a \leq k \leq b$.

(a) $a = -13$, $b = 7$ (b) $a = -15$, $b = 10$ (c) $a = -13$, $b = 5$

(d) $a = -15$, $b = 5$ (e) none of these

22. What is the sum of the y-components of the real ordered pair solutions (x, y) of the system of equations

$$\begin{aligned} 4^{xy+4} &= 8^{x^2-y-1} \\ y &= x+1 \ ? \end{aligned}$$

(a) 5 (b) 6 (c) 7 (d) 8 (e) none of these

23. An observer spots an airplane that is flying horizontally on a course that will take it directly overhead in a few minutes. The plane is known to cruise at 540 miles per hour. On the initial sighting the angle of elevation was 20°, and 40 seconds later it was 27°. What is the altitude of the plane, in miles?

(a) $\dfrac{360 \sin 153^\circ \sin 20^\circ}{\sin 7^\circ}$ (b) $\dfrac{360 \sin 153^\circ \sin 20^\circ}{\sin 27^\circ - \sin 20^\circ}$

(c) $\dfrac{6 \sin 153^\circ \sin 20^\circ}{\sin 27^\circ - \sin 20^\circ}$ (d) $\dfrac{6 \sin 153^\circ \sin 20^\circ}{\sin 7^\circ}$ (e) none of these

24. Which of the following six statements are true about the cubic polynomial $P(x) = 2x^3 + x^2 + 3x - 2$?

(i) It has exactly one positive real root.

(ii) It has either one or three negative roots.

(iii) It has a root between 0 and 1.

(iv) It must have exactly two real roots.

(v) It has a negative root between -2 and -1.

(vi) It has no complex roots.

(a) only (i) (b) only (i), (iii) and (vi) (c) only (ii), (iii) and (iv)

(d) only (i) and (iii) (e) only (iii), (iv) and (v)

25. Find the largest positive integer that has the property that if you divide it into each of 4201, 6301, and 7351 the remainder will be 1.

(a) 1050 (b) 2100 (c) 50 (d) 150 (e) none of these

26. A triangle has one vertex at $(0,0)$ and the other two on the graph of $y = -2x^2 + 54$, at (x, y) and $(-x, y)$ where $0 < x < \sqrt{27}$. Find x such that the corresponding triangle has maximum area.

(a) $\sqrt{27}/2$ (b) 3 (c) $\sqrt{3}$ (d) $2\sqrt{3}$ (e) none of these

27. The line $\mathcal{L}$ is tangent to the curve $x^2y = 8$ at the point $(2, 2)$. Compute the area below $\mathcal{L}$ in the first quadrant.

(a) 9 (b) 8 (c) 1 (d) $4\sqrt[3]{2}$ (e) $4\sqrt{2}$

28. Which of the following is always true about a function $f(x)$ on the interval $[a, b] = \{x \mid a \leq x \leq b\}$?

(a) If $f(x) \geq 0$ on $[a, b]$, then $\displaystyle\int_a^b f(x)dx \leq \int_a^b f^2(x)dx$

(b) If $f(x)$ is increasing on $[a, b]$, then $f^2(x)$ is increasing on $[a, b]$

(c) If $f(x)$ is increasing on $[a, b]$, then $f(x) \geq 0$ on (a, b)

(d) If $f(x)$ attains a minimum at c where $a < c < b$, then $f'(c) = 0$

(e) none of these

29. An equation for the line that both passes through (10,-1) and is perpendicular to $y = \frac{1}{4}x^2 - 2$ is:

(a) $4x + y = 39$ (b) $2x + y = 19$ (c) $x + y = 9$ (d) $x + 2y = 8$ (e) none of these

30. Figure $ABCDE$ is a regular pentagon with all sides equal to 4. Which of the following is (are) a correct solution(s) for the length of AC?

(i) $2\csc(18^\circ)$ (ii) $2\sec(72^\circ)$ (iii) $\sqrt{32 - 32\cos(108^\circ)}$

(a) only (i) is correct (b) only (ii) is correct (c) only (iii) is correct

(d) only (i) and (ii) are correct (e) all are correct